ESSENTIALS OF

CALCULUS

BY

E. J. TOWNSEND, Ph.D. (Göttingen)
Professor of Mathematics
University of Illinois

AND

G. A. GOODENOUGH, M.E. (Illinois)
Associate Professor of Mechanical Engineering
University of Illinois

SECOND EDITION, REVISED

NEW YORK
HENRY HOLT AND COMPANY

6

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

In the preparation of this volume, the authors have had in mind the needs of those colleges and technical schools in which the time devoted to calculus is limited to a three-hour course for a year, or perhaps to a five-hour course for two terms.

The usual division of the subject into differential and integral calculus has been largely disregarded. By the arrangement adopted, the student is early led by easy steps into simple practical applications of the calculus; and the more difficult topics are postponed until late in the course.

The theory of limits has been used exclusively in the development of fundamental principles. Throughout the book much emphasis is placed upon the applications of the calculus to practical problems. Only such knowledge of physics on the part of the student is assumed as is usually included in an elementary course in that subject. Some problems are introduced that show the use of calculus in discussing well-known applications to physical and engineering phenomena. Such problems are so stated, however, as to require no technical knowledge on the part of the student. The applications to geometry are such as are essential and usually to be found in a first course in calculus.

In the selection of material, the authors have departed somewhat from the traditional course. Many topics usually included in calculus have been entirely omitted or greatly reduced in extent. Thus, but little attention has been given to special methods of integration; and reduction formulas for integration, order of contact, envelopes, etc., have been omitted entirely. On the other hand, some parts of the text have been extended beyond the usual limits. Functions of two or more variables, because of their importance in physics, have been discussed more fully than usual. Special stress has been laid on the summation

process, and by numerous examples from physics the student is drilled in the choice of proper elements and in the setting up of definite integrals. Attention is called to the treatment of exact and inexact differentials; a subject of first importance in the physical applications of calculus, but one that usually receives little or no consideration.

The authors take this occasion to express their obligations to their colleagues at the University of Illinois and elsewhere for their helpful suggestions in the preparation of this book, and to the publishers for their coöperation in making its typography of high grade. The authors are under special obligations to Professor H. L. Rietz and to Dr. A. R. Crathorne for their assistance in seeing the book through the press.

E. J. TOWNSEND.
G. A. GOODENOUGH.

UNIVERSITY OF ILLINOIS,
July, 1910.

CONTENTS

CHAPTER I

CONSTANTS, VARIABLES, FUNCTIONS

CHAPTER II

DERIVATIVES OF ALGEBRAIC FUNCTIONS

CHAPTER III

ELEMENTARY APPLICATIONS OF DERIVATIVES

CHAPTER IV

THE DIFFERENTIAL NOTATION

CHAPTER V

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

CHAPTER VI

INTEGRATION

CHAPTER VII

SIMPLE APPLICATIONS OF INTEGRATION

CHAPTER VIII

SUCCESSIVE DIFFERENTIATION AND INTEGRATION

CHAPTER IX

CURVES

CHAPTER X

DEFINITE INTEGRALS

CHAPTER XI

APPLICATIONS OF INTEGRATION TO GEOMETRY AND MECHANICS

CHAPTER XII

SPECIAL METHODS OF INTEGRATION

CHAPTER XIII

FUNCTIONS OF TWO OR MORE VARIABLES

CHAPTER XIV

MULTIPLE INTEGRALS, APPLICATIONS

CHAPTER XV

INFINITE SERIES

COURSE IN CALCULUS

GREEK ALPHABET

Letters		Names	Letters		Names
Capitals	Lower Case		Capitals	Lower Case	
Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ε	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

CALCULUS

CHAPTER I

FUNDAMENTAL NOTIONS AND DEFINITIONS

1. Constants, variables. In the applications of mathematics to physical problems, we meet with such magnitudes as velocity, force, mass, length, area, volume, etc. The measure of a magnitude is expressed by means of a number, represented by a figure or letter, which denotes the ratio that the given magnitude bears to some standard magnitude of the same kind adopted as a unit. For the purposes of calculation, it is the number that is of fundamental importance. For the sake of brevity, however, we shall often speak of "a velocity v" or "an area A," etc., instead of using the longer but perhaps more precise expressions "a velocity whose measure is v units" or "an area whose ratio to the standard unit of area is A," etc.

In any particular discussion, the magnitudes and consequently the corresponding numbers may or may not change. A number that remains unchanged is called a **constant**. A symbol is then said to represent a constant when it denotes but one value in a given discussion. A symbol that satisfies this condition is itself often called a constant.

When a number is permitted to assume different values in the same discussion, it is called a **variable**. A symbol is said to represent a variable if it has assigned to it different values in the same discussion. Here again it is usual to speak of the symbol itself as the variable.

For example, suppose a body falls from rest. The law that gives the relation between the time t and the distance s through which the body falls is expressed by the equation

$$s = \tfrac{1}{2} gt^2. \tag{1}$$

Here the number g is a constant denoting the acceleration at a point on the earth's surface. On the other hand, the time t and the distance s are variables, whose mutual relation is determined by the given equation.

Again, in the equation of the circle

$$(x-2)^2+(y-3)^2=r^2, \tag{2}$$

2 and 3 are constants that determine the center of the circle, and r is a constant denoting the length of the radius. However, x and y are variables whose corresponding values, as determined by the given equation, fix the various positions of the generating point. To the constant r we may assign any value at pleasure, but when it has been once assigned it must remain unchanged, so long as the circle remains fixed. A constant of this character is called an **arbitrary constant** or **parameter.**

2. Functions. In each of the illustrations given in the preceding article, it will be observed that the variables involved stand in an intimate relation to each other. For example, in the law of falling bodies, s and t are so related that to any value assigned to t there corresponds a definite value of s. Moreover, to every positive value that we may give s, the given relation between s and t determines corresponding values of t. Again, in the equation of the circle the variables x and y are related in a similar manner. Other illustrations of such relations between variables are familiar to the student from his study of elementary mathematics and physics. Whenever such a relation exists between two variables, we say that the variables are connected by a functional relation, or that the one is a function of the other. We may then define a function as follows:

If two variables are so related that for each value that may be assigned to the one there are determined one or more definite values of the other, the second variable is said to be a **function** *of the first.*

The variable to which we may assign arbitrarily chosen values is called the **independent variable.** It is also frequently referred to simply as the variable or argument. The variable which is thus determined, that is, the function, is sometimes called the **dependent variable.**

A function may depend for its value upon two or more independent variables. For example, the area of an ellipse is a function of the lengths of the major and the minor axes; the volume of a gas is a function of the temperature and of the pressure to which the gas is subjected. Such functions are said to be functions of two variables. For the present we shall consider only functions of a single variable; later, some of the properties of functions of two or more variables will be discussed.

3. Fundamental problems of the calculus. In his study of elementary mathematics and in his everyday experiences, the student has frequently had occasion to deal with magnitudes that change. In many instances the changes are abrupt and sometimes periodic. For example, the market price of any commercial product changes abruptly from time to time. When money is placed at compound interest, the amount of interest is usually added to the principal at certain intervals. Among physical phenomena, on the other hand, we encounter numerous illustrations of changes which are evidently continuous. Thus the pressure of the atmosphere varies with the altitude, but the change is gradual, not abrupt; the speed of a railway train starting from a station changes continuously until the maximum speed is reached; the pressure of a liquid upon a vertical wall increases continuously with the depth. Many other illustrations of both continuous and discontinuous changes will occur to the student.

Problems involving discontinuous changes are dealt with for the most part by the ordinary processes of arithmetic and algebra. Problems that involve continuous changes require more powerful mathematical methods and these are the special province of the calculus. To illustrate the methods of the calculus and in a general way give the student some notion of the scope of the subject and of the class of problems to be considered, a few typical problems of fundamental importance are here introduced.

4. Problem I. Slope of a tangent. Required the angle that the tangent to a given curve at a point P_1 (Fig. 1) makes with the X-axis. To make the problem concrete let the equation of the curve be $y = 3\sqrt{x}$, and let the abscissa of P_1 be $x = 4$.

Through the given point P_1 let a secant line be drawn cutting

the curve in a second point P. Denoting by Δx the increase P_1Q, in the abscissa between P_1 and P, by Δy the corresponding increase

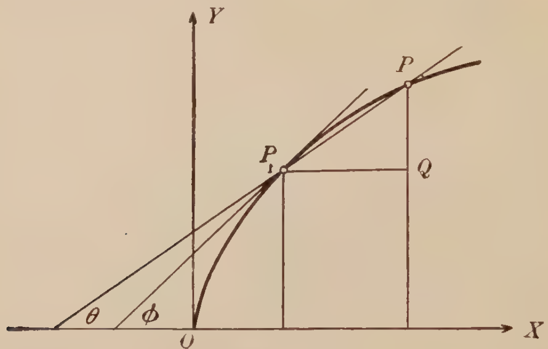

FIG. 1.

QP of the ordinate, and by θ the angle that the secant makes with the X-axis, we have

$$\tan\theta = \frac{QP}{P_1Q} = \frac{\Delta y}{\Delta x}.$$

As the point P is made to approach the fixed point P_1, the secant approaches as a limiting position the tangent at P_1 and the angle θ approaches the angle ϕ. Hence, by taking Δx smaller and smaller and calculating the ratio $\frac{\Delta y}{\Delta x}$ for the various values of Δx, we can approximate more and more closely the value of $\tan\phi$. The following table gives the calculated values for different assumed values of Δx.

Δx	Δy	$\frac{\Delta y}{\Delta x}$
2.	1.34847	0.67424
1.	0.708204	0.70820
0.1	0.074537	0.74537
0.01	0.007495	0.74953
. .	. . . .	. . .

Using this arithmetical method, we can arrive at an approximate value of $\tan\phi$. The method, however, is tedious and the result is

at best an approximation. What we need is the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx approaches the value zero. This limit can be found by considering the equation of the curve.

Let (x_1, y_1) be the coördinates of the given point P_1. The coördinates of the point P are then $(x_1 + \Delta x, y_1 + \Delta y)$. Hence from the equation of the curve, $y = 3\sqrt{x}$, we have

$$y_1 = 3\sqrt{x_1}, \tag{1}$$

$$y_1 + \Delta y = 3\sqrt{x_1 + \Delta x}. \tag{2}$$

By subtraction, we have

$$\Delta y = 3(\sqrt{x_1 + \Delta x} - \sqrt{x_1}), \tag{3}$$

and therefore it follows that

$$\frac{\Delta y}{\Delta x} = 3\frac{\sqrt{x_1 + \Delta x} - \sqrt{x_1}}{\Delta x}. \tag{4}$$

Rationalizing the numerator, we obtain

$$\frac{\Delta y}{\Delta x} = \frac{3\,\Delta x}{\Delta x(\sqrt{x_1 + \Delta x} + \sqrt{x_1})} = \frac{3}{\sqrt{x_1 + \Delta x} + \sqrt{x_1}}. \tag{5}$$

As Δx approaches zero, the expression $\frac{3}{\sqrt{x_1 + \Delta x} + \sqrt{x_1}}$ approaches $\frac{3}{2\sqrt{x_1}}$, and since at the same time $\tan\theta$, which is given by the ratio $\frac{\Delta y}{\Delta x}$, approaches $\tan\phi$, we infer that $\tan\phi = \frac{3}{2\sqrt{x_1}}$. We have then the result that for $x_1 = 4$, $\tan\phi = \frac{3}{2\sqrt{4}} = 0.75$.

The method here developed has the added advantage, that the result is general and the slope of the tangent at any point can be found when once we know the abscissa of the point. All we need to do is to substitute the given value of x in the general formula $\frac{3}{2\sqrt{x}}$. The difficulty of applying this method to all problems is that in most cases that arise it is not easy to find the limiting value of $\frac{\Delta y}{\Delta x}$. In the subsequent chapters of the calculus we shall

develop methods by means of which this limit, known as the **derivative** of y with respect to x, may be obtained directly from the given functional relation between x and y.

5. Problem 2. Speed of a falling body. To determine the *average speed* of a moving body during a given time, we simply divide the distance traveled by the time occupied. Thus, an eighteen-hour train from Chicago to New York has an average speed of $905 \div 18 = 50\frac{5}{18}$ miles per hour. The speed at any particular instant is not so easily determined. The obvious way to find this speed approximately is to take small intervals of time, say 5 seconds, 1 second, $\frac{1}{10}$ second, etc., immediately following the instant in question, and divide the distance traversed in the assumed time interval by that time interval. The result is the mean speed for the time interval, and it is evident that the smaller the chosen interval is taken the more nearly the quotient gives the instantaneous speed at the beginning of the interval. Such a method is, however, open to the criticism mentioned in the previous problem; namely, no matter how small the interval taken, the result is merely an approximation. To obviate this objection we make use of the definite law which the motion follows and as in problem 1 find the value of the limit involved.

As a concrete case, let us consider the motion of a body falling in a vacuum. From physics, it is known that $s = \frac{1}{2}gt^2$, where s denotes the distance traversed in t seconds starting from rest, and g is a constant whose value is approximately 32.2. Suppose we wish to find the speed at the end of t_1 seconds. The distance s_1 traversed in the time t_1 is given by

$$s_1 = \tfrac{1}{2}gt_1^2. \tag{1}$$

If Δt denotes the assumed time interval immediately following t_1, and Δs denotes the distance traversed in this interval of time, we have

$$s_1 + \Delta s = \tfrac{1}{2}g(t_1 + \Delta t)^2. \tag{2}$$

From (1) and (2) we obtain

$$\begin{aligned}\Delta s &= \tfrac{1}{2}g[(t_1 + \Delta t)^2 - t_1^2] \\ &= gt_1\Delta t + \tfrac{1}{2}g(\Delta t)^2.\end{aligned}$$

Therefore, we have $$\frac{\Delta s}{\Delta t} = gt_1 + \tfrac{1}{2}g\,\Delta t. \tag{3}$$

Equation (3) gives the mean speed for the time Δt, and it is seen that this speed is a variable magnitude depending upon the assumed time interval Δt.

As Δt is taken smaller, this mean speed for the interval approaches the instantaneous speed at the end of t_1 seconds. However, as Δt becomes smaller, the expression $gt_1 + \frac{1}{2}g\,\Delta t$ approaches the fixed value gt_1. We conclude, therefore, that gt_1 is the actual speed at the end of t_1 seconds. At the end of 3 seconds, for example, the speed is $32.2 \times 3 = 96.6$ feet per second, and at the end of 10 seconds 322 feet per second. When we have learned how to find the derivative of the function s with respect to t, that is, the limit of the ratio $\frac{\Delta s}{\Delta t}$, directly from the functional relation between s and t, the process here indicated will be very much simplified.

6. Problem 3. Given a derivative, to find the original function. In problems 1 and 2 we saw the importance of being able to find from the functional relation between two variables the derivative of the one with respect to the other, that is, the limiting values of $\frac{\Delta y}{\Delta x}$ and $\frac{\Delta s}{\Delta t}$. In the one case, the result gave the slope of the tangent to a curve, and in the other, the speed of a moving body at a particular instant. It is often equally as important to be able to solve the inverse problem; that is, if we have given $\frac{3}{2\sqrt{x}}$ as the slope of the tangent to a curve, or gt as the speed of a moving body, it is of value in certain discussions to be able to say that the curve in question is given by the functional relation $y = 3\sqrt{x}$ or in the other case that the law of motion for the body is expressed by the equation $s = \frac{1}{2}gt^2$. A process of the calculus known as integration enables us to solve such problems.

7. Problem 4. Area under a curve. One of the most important problems in calculus, in fact, one of the problems that led to the invention of the calculus, is that of finding the area between a given curve, the X-axis and two given ordinates. To take a con-

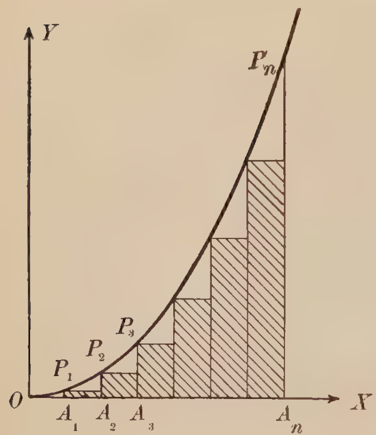

FIG. 2.

crete example, let us attempt to find the area between the parabola $y = x^2$, the X-axis, the origin, and the ordinate $x = 3$. See Fig. 2.

Let the part of the X-axis between $x = 0$ and $x = 3$ be divided into n equal parts, each denoted by Δx. At the subdivisions A_1, A_2, etc., let ordinates be erected and rectangles be constructed as shown in the figure. The abcissas of the points A_1, A_2, A_3, etc., are Δx, $2\,\Delta x$, $3\,\Delta x$, etc., and the altitudes of the successive rectangles are 0, $(\Delta x)^2$, $(2\,\Delta x)^2$, etc. Hence the sum of the areas of the rectangles is

$$\begin{aligned} A_r &= 0 \cdot \Delta x + \Delta x\,(\Delta x)^2 + \Delta x\,(2\,\Delta x)^2 + \cdots + \Delta x\,(\overline{n-1}\,\Delta x)^2 \\ &= (\Delta x)^3[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]. \end{aligned} \tag{1}$$

From algebra,* we have

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \frac{n\,(n-1)(2\,n-1)}{6}. \tag{2}$$

Hence, we have from (1)

$$\begin{aligned} A_r &= (\Delta x)^3\,\frac{n\,(n-1)(2\,n-1)}{6} \\ &= n \cdot \Delta x \left\{\frac{2\,(n \cdot \Delta x)^2 - 3\,(n \cdot \Delta x)\,\Delta x + (\Delta x)^2}{6}\right\}. \end{aligned} \tag{3}$$

But, it will be remembered that

$$n \cdot \Delta x \equiv \text{segment } OA_n = 3.$$

Equation (3) then becomes

$$A_r = 3\left(\frac{18 - 9\,\Delta x + (\Delta x)^2}{6}\right). \tag{4}$$

---

* Rietz & Crathorne's *College Algebra*, p. 87, Ex. 4.

Equation (4) gives the sum of the areas of the rectangles for any assumed value of Δx. As Δx is taken smaller and smaller and the number of rectangles is correspondingly increased, the area A_r approaches as a limit the area A under the curve, that is, the area OP_nA_n. However, from (4) it follows that as Δx approaches zero, A_r has the limiting value $\frac{3 \times 18}{6} = 9$, and hence we have as a result $A = 9$.

This problem illustrates a large and important class of problems considered in calculus, namely, those requiring for their solution the summation of an indefinitely large number of indefinitely small elements. All problems in areas and volumes, and many of the applications to mechanics and mathematical physics, are of this character.

The preceding problems have been selected as typical of the kind of problems to be discussed later and as indicating somewhat the class of problems to which the methods of the calculus particularly apply. The student will have observed that each of these problems depends for its solution upon the finding the limiting value of some function. However, the methods employed in these examples for finding the limit are not of great value, because they become too complicated and tedious when applied to any but the simplest cases. For example, in the last problem, we were able to find the limit easily because the sum of the squares of the first n integers happens to be known. If the curve were given by the equation $y = \sqrt{x}$, we should have

$$A_r = (\Delta x)^{\frac{3}{2}}(\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n-1}),$$

and the solution would be much more difficult. In the succeeding chapters we shall develop methods for accomplishing this same purpose much more directly and easily. In the meantime it is essential that we call attention to some of the fundamental properties of functions and the laws of limits as applied to the problems that we shall need to consider.

EXERCISES

1. By the method of problem 4 find the area between the X-axis, the line $y = 2x$, and the ordinate $x = 5$.

2. Find the slope of the tangent to the curve $y = x^2$ at the point $x = 3$.

3. Find the slope of the tangent to the curve $y = \frac{12}{x}$ at the point $x = 6$

4. If a body is projected vertically upward with an initial speed v_0, the relation between the distance s and the time t is $s = v_0 t - \frac{1}{2} g t^2$. Find an expression for the speed at the end of t_1 seconds. If $v_0 = 150$, find the speed at the end of 4 seconds.

8. Functional notation. As we have seen (Arts. 2, 4, 5), the relation between a variable and a function of it is expressed by an equation; as $y = 3\sqrt{x}$, $s = \frac{1}{2} g t^2$. However, even when the analytic relation between the independent and dependent variables is known, it is convenient to have a general symbol that shall stand for the function as expressed in terms of its variable. For this purpose the variable is inclosed in a parenthesis and some letter is prefixed. Thus, a function of x may be denoted by such symbols as the following:

$$f(x),\ F(x),\ \phi(x),\ \psi(x), \text{ etc.},$$

which are read "f function of x" (or simply "f of x"), "F function of x," etc. It is to be understood that the letters f, F, ϕ, ψ, etc., are symbols of functionality and not factors. Whenever two or more different functions are employed in the same discussion, a different symbol must be used for each.

If a function depends upon two variables instead of one, it may be expressed symbolically by $f(x, y)$, $\phi(x, y)$, etc.

When, in connection with any discussion, we have once defined $f(x)$, then $f(a)$ denotes the same expression with x replaced by a. In a similar way we may form functions $f(t)$, $f(4)$, $f(x+h)$, etc. Thus, if we have

$$f(x) = 3x^2 + 7x + 9,$$

it follows that

$$f(t) = 3t^2 + 7t + 9,$$
$$f(4) = 3 \cdot 4^2 + 7 \cdot 4 + 9 = 85,$$
$$f(x+h) = 3(x+h)^2 + 7(x+h) + 9.$$

Instead of replacing the variable by another variable or by a constant, we may replace it by any function of the same variable or of a different variable. Thus we may write

$$f[\phi(x)],\ F[f(y)],\ \psi(\sin\theta), \text{ etc.}$$

Given, for example, the function

$$F(x) = x^2 - 3x + 7,$$

we have

$$F(\sin x) = \sin^2 x - 3 \sin x + 7.$$

EXERCISES

1. Given $f(x) = x^3 - 6x^2 + 5x - 14$. Find the values of $f(4)$, $f(0)$, $f(-3)$. Write the expression for $f(\sin\theta)$ and for $f(x+h)$.

2. If $\phi(x) = 5x^2 - 2x + 8$, write expressions for $\phi(x^2)$, $\phi(-x^3)$, $\phi(\tan\theta)$, $\phi(0)$, $\phi(t)$.

3. If $F(\theta) = \cos\theta$, find the value of $F(0)$, $F\left(\frac{\pi}{4}\right)$, $F\left(\frac{\pi}{2}\right)$, $F(\pi)$.

4. If $f(x) = x^5 - 4x^3 + x$, show that $f(-x) = -f(x)$.

5. If $f(x) = x^4 - 6x^2 + 1$, show that $f(-x) = f(x)$.

6. Given $F(x) = 3 - \sqrt{x}$ and $f(y) = y^2 + 4$. Find $F[f(y)]$ and $f[F(x)]$, also $f[f(y)]$ and $F[F(x)]$.

7. Given $y = f(x) = \dfrac{1+x}{1-x}$, and $z = f(y) = f[f(x)]$; express z as a function of x.

8. If $\phi(x) = 2x^2 - 1$, show that $\phi(\cos\theta) = \cos 2\theta$.

9. If $\phi(x) = 2x\sqrt{1-x^2}$, show that $\phi(\sin\theta) = \phi(\cos\theta) = \sin 2\theta$.

10. If $f(x) = \log_a x$, show that $f(x) - f(y) = f\left(\dfrac{x}{y}\right)$, and $f(x) + f(y) = f(xy)$.

11. If $f(\theta) = \cos\theta$, show that $f(\theta) = f(-\theta) = -f(\pi+\theta) = -f(\pi-\theta)$.

12. If $f(x) = \sin x$ and $\phi(x) = \cos x$, find $f(x)\cdot\phi(y) \pm \phi(x)\cdot f(y)$.

13. If $f(x) = e^x$,* show that $f(x)\cdot f(y) = f(x+y)$.

14. If $f(x) = \sqrt{1-x^2}$, find $f(\sin\theta)$ and $f(\cos\theta)$.

15. If $f(x) = \sqrt{1+x^2}$, find $f(\tan\theta)$.

16. If $f(x) = \tan x$, find $\dfrac{f(x) - f(y)}{1 + f(x)\,f(y)}$.

9. Graphs of functions. In accordance with the principles of analytic geometry, a geometric interpretation may be given to

---

* The symbol e is used to denote the base of the natural system of logarithms, namely, 2.718 $\cdots$. Log x indicates the logarithm of x to the base e. When any other base is used, that fact will be indicated by a subscript, as $\log_a x$, $\log_{10} 30$.

the relation of a function to its variable. If the values of the independent variable be laid off as abscissas, the corresponding values of the function may be used as ordinates. The assemblage of the extremities of these ordinates is called the **graph** of the function. It is to be noted that the ordinates of the points on the graph and not the graph itself represent the values of the function. It is assumed that the student is already familiar with the graphical representation of functions from his study of analytic geometry.

10. Definition of a limit. The general notion of a limit is perhaps fairly clear from the illustrative problems, Arts. 4–7. We must, however, formulate a definition with sufficient accuracy that it may be made the basis of a mathematical investigation. For this purpose, let us consider two variables, one a function of the other; say, $y = f(x)$.

Let the independent variable x vary in such a manner that it differs less and less numerically from some constant a, that is, in such a way that the numerical value of $x - a$, written $|x - a|$, becomes and remains less than any positive constant however small. We say then that x approaches a, and indicate that fact by writing $x \doteq a$, which is to be read "x approaches a."

As the independent variable x approaches the constant a the dependent variable or function $f(x)$ assumes a corresponding set of values. When x becomes very nearly equal to a, these values of $f(x)$ may at the same time become very nearly equal to some constant, say A. Moreover, it may occur that we may not only make $f(x)$ differ as little as we please from A by taking a value of x sufficiently near to a, but that it will remain at least as close to A for all values of x that lie between a and this chosen value of x. When these conditions are fulfilled, we call A the **limit** of the function $f(x)$ as x approaches a.

The problem of falling bodies, Art. 5, furnishes a good illustration. The function $gt_1 + \frac{1}{2} g \Delta t$ not only may be made to differ as little as we please from gt_1 by taking Δt sufficiently small, but as Δt assumes any value still closer to zero the function differs less from gt_1. Hence we were justified in speaking of gt_1 as the limit of $gt_1 + \frac{1}{2} g \Delta t$, as $\Delta t \doteq 0$.

We may now define a limit as follows:

If a constant A can be found such that, as x approaches a, $f(x) - A$ becomes and remains less numerically than any constant however small, then A is called the limit of $f(x)$ as x approaches a.

If the function $f(x)$ has the limit A, we indicate the fact by writing

$$\underset{x \doteq a}{L} f(x) = A. \tag{1}$$

Since by the definition of a limit $f(x)$ can be made to differ from A as little as we please by the proper selection of x, we may also write

$$f(x) = A + |\epsilon(x)|, \tag{2}$$

where $\epsilon(x)$ can be made as small as we choose by taking x sufficiently near to a. Equations (1) and (2) are equivalent and are merely different forms of expression for the same relation. In some demonstrations form (2) will be found the more convenient.

Ex. Consider the limit of the function

$$f(x) = \frac{4}{x^2+1}$$

as $x \doteq 2$. By taking the value of x sufficiently near 2, we may make the value of $f(x)$ differ as little as we please from $\frac{4}{5}$. Hence, we have

$$\underset{x \doteq 2}{L} \frac{4}{x^2+1} = \frac{4}{5}.$$

In the preceding discussion of limits nothing has been said about the value of $f(x)$ for $x = a$. The limit depends for its value upon the values of $f(x)$ for x very close to $x = a$, or, as we frequently say, "in the neighborhood of a," and it is not affected by the value that $f(x)$ takes for $x = a$. As we shall see later the value that the function takes for $x = a$, that is, $f(a)$, may or may not be equal to the limit of $f(x)$ as x approaches a.

To determine whether a function has a limit as the variable approaches a definite value, we must consider *all* values of the function in the neighborhood of the limiting value of the variable. In other words, a limit of the values of the function must exist as the variable approaches its limit from either direction, and

these two limiting values of the function must be equal. The following example serves as an illustration.

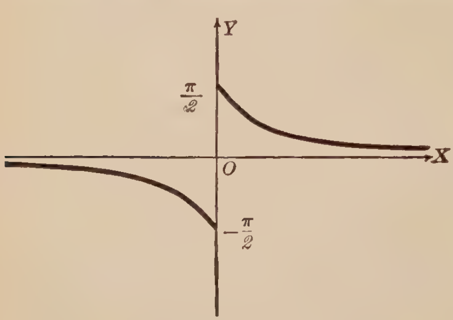

FIG. 3.

Ex. Given the function $y = \text{arc}\tan\frac{1}{x}$, the graph of which is shown in Fig. 3. When the independent variable is restricted to positive values, we obtain the limit $\frac{\pi}{2}$ as $x \doteq 0$; when negative values only are considered, we get $-\frac{\pi}{2}$ as $x \doteq 0$. Hence the function cannot be said to have a limit as $x \doteq 0$.

11. Infinity. In the discussion of limits thus far, we have considered only the case in which the independent variable approaches a definite number. The student is familiar from his study of algebra and geometry with the case in which the independent variable increases without limit. For example, the area of a circle is the limit of the area of a regular inscribed polygon as the number of sides increases without limit. Again, the value of an infinite series is obtained by taking the limit of the sum of a finite number of terms as the number of terms increases without limit. We say in such cases that the independent variable **becomes infinite**, and express that fact by writing $x = \infty$, $n = \infty$, etc. Here the symbol $=$ is not to be understood in the ordinary sense of "is equal to," but rather in the sense of "becomes," and the above expressions should be read "x becomes infinite," "n becomes infinite," etc. Instead of the independent variable, the function may become infinite. This may occur when the variable also becomes infinite, or when it approaches a definite number. Thus x^{-1} becomes infinite as $x \doteq 0$. While it is customary to write $\underset{x \doteq 0}{L}\, x^{-1} = \infty$, the function cannot be said to have a limit, because it does not approach any *definite* number however large that number may be.

When a function has the limit zero, it is often spoken of as an **infinitesimal.**

The introduction of the concept infinity extends the use of limits so as to include a large number of important applications, of which the following are illustrations.

Ex. **1.** An air pump is used to remove air from an inclosed space. At each stroke of the piston, part of the air is removed, and in consequence the density of the air remaining is diminished. Let d denote the original density, d_n the density after n strokes of the piston, v the volume of the inclosed space, and v' the volume of the pump cylinder. From physics, we have

$$d_n = d\left(\frac{v}{v+v'}\right)^n = \frac{d}{\left(1+\frac{v'}{v}\right)^n}.$$

Here the density d_n is a function of the number of strokes. If n is increased without limit, we have, since the denominator is greater than unity,

$$\underset{n=\infty}{L} \frac{d}{\left(1+\frac{v'}{v}\right)^n} = 0.$$

Hence, as the variable increases without limit, the function approaches 0 as a limit. With a perfect pump we may therefore make the density as small as we please by taking the number of strokes sufficiently large.

Ex. **2.** The work done by a gas in expanding from a volume v_1 to a volume v according to the law $pv^k =$ const. is given by the expression

$$\frac{p_1 v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}} - \frac{1}{v^{k-1}}\right].$$

If the expansion proceeds without limit, that is, if $v = \infty$, the expression for the work becomes

$$\underset{v=\infty}{L} \frac{p_1 v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}} - \frac{1}{v^{k-1}}\right] = \frac{p_1 v_1^k}{k-1}\left[\frac{1}{v_1^{k-1}} - 0\right] = \frac{p_1 v_1}{k-1}.$$

In this case the function approaches a definite limit as the variable increases without limit.

Ex. **3.** If the gas expands according to Boyle's law, $pv =$ const., the expression for the work done is $p_1 v_1 \log \frac{v}{v_1}$. If, in this case, the volume v increases without limit, we have for the work done

$$\underset{v=\infty}{L}\ p_1 v_1 (\log v - \log v_1) = \infty;$$

that is, the function representing the work done also increases without limit.

12. Continuity of functions. We have seen that the value of a limit depends upon the values of the function in the vicinity of

the limiting value of the variable, and not upon the value of the function at that point. In some cases the limit is the same as the value of the function at the limiting point, and in some cases it is not. If the two values are equal, the function is said to be **continuous** at the point in question. If, on the other hand, the limit is different from the value of the function at the limiting point, or if for any reason the function has no limit, the function is said to be **discontinuous** at that point. The condition that $f(x)$ shall be continuous for $x=a$ is then given by the equation

$$\underset{x \doteq a}{L}\, f(x) = f(a).$$

As we shall see, continuity is a very important property of the functions to be discussed later. In fact, we shall confine ourselves for the most part to functions having this property. The following examples will serve to make clear the distinction between continuous and discontinuous functions.

Ex. **1.** The function

$$s = f(t) = \tfrac{1}{2} g t^2$$

is continuous for all values of t; for, we have

$$\underset{t \doteq t_0}{L}\, \tfrac{1}{2} g t^2 = \tfrac{1}{2} g t_0^2 = f(t_0),$$

where t_0 is any definite value of t.

Ex. **2.** Consider the function

$$f(x) = \frac{1}{x},$$

for values in the neighborhood of the origin. We have

$$\underset{x \doteq 0}{L}\, \frac{1}{x} = \infty;$$

which is another way of saying that as $x \doteq 0$, the value of the function increases without limit, and hence has no limit. The function is therefore discontinuous for $x=0$. A function always has a discontinuity for any value of the variable for which the function becomes infinite.

We have a similar condition in Boyle's law, which is expressed by the equation $pv = k$ or $p = k\dfrac{1}{v}$. If the volume v be diminished indefinitely, the pressure increases without limit and hence has a discontinuity for $v = 0$.

EX. 3. Suppose a given mass of ice at a temperature below 32° F. to be heated. As heat is applied, the temperature of the ice rises, and the quantity of heat Q is a function of the temperature τ. So long as τ remains below the melting point the function is continuous. But when the melting point 32° F. is reached, a quantity of heat represented by AB, Fig. 4, is absorbed without any change in temperature. Likewise, when the water reaches the boiling point, heat represented by CD is absorbed without change of temperature. At these particular temperatures the function is therefore discontinuous.

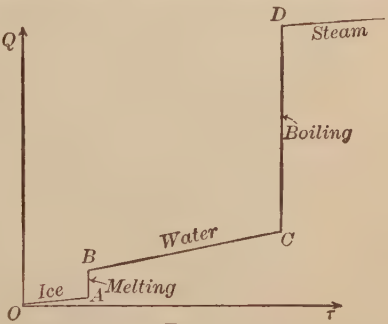

FIG. 4.

In the preceding discussion, we have spoken only of the continuity of a function at a single point. If a function is continuous at all points of an interval, then it is said to be **continuous throughout the interval.**

In plotting a graph, it is of great assistance to know that the graph represents a continuous function and has a definite direction at each point; for then we need only to locate points of the graph sufficiently dense to give the general outline and to draw through these points a continuous curve. Since it is obviously impossible to locate all of the points of a curve, this is, in fact, the only way in which a graph can be drawn. In case the function has a discontinuity, care must be taken to locate points sufficiently dense in the neighborhood of the discontinuity to determine its character.

EXERCISES

1. Which of the trigonometric functions have points of discontinuity ?

2. Find the values of x for which the function $\dfrac{7x+9}{x^2-5x+6}$ is discontinuous.

3. If $f(x)$ and $\phi(x)$ are integral rational functions, under what conditions is the function $\dfrac{f(x)}{\phi(x)}$ discontinuous ?

4. Examine for continuity the functions

$$\text{(a) } y = \operatorname{arc}\tan\frac{1}{x}; \qquad \text{(b) } y = \frac{1}{1+e^{\frac{1}{x}}}.$$

13. Laws of operation with limits. In this section, we shall state without proof * several properties of limits that will be utilized in later discussions. These are given in the following theorems.

Theorem I. *The limit of a constant times a function is equal to the constant times the limit of the function ; that is,*

$$\underset{x \doteq a}{L}\, c \cdot f(x) = c \cdot \underset{x \doteq a}{L}\, f(x).$$

Theorem II. *If each of two functions approaches a limit, the limit of the sum (or difference) of the functions is equal to the sum (or difference) of the limits; that is,*

$$\underset{x \doteq a}{L}\, [f(x) + \phi(x)] = \underset{x \doteq a}{L}\, f(x) + \underset{x \doteq a}{L}\, \phi(x).$$

Theorem III. *If each of two functions has a limit, the limit of the product of the functions is equal to the product of the limits; that is,*

$$\underset{x \doteq a}{L}\, [f(x) \cdot \phi(x)] = \underset{x \doteq a}{L}\, f(x) \cdot \underset{x \doteq a}{L}\, \phi(x).$$

When Theorem III is extended to the product of a finite number n of *equal* functions, we have the following:

Corollary. *If a function has a limit, the limit of the nth power of the function is equal to the nth power of its limit.*

Theorem IV. *If each of two functions has a limit, the limit of the quotient of the functions is equal to the quotient of the limits, provided the limit of the denominator is different from zero; that is,*

$$\underset{x \doteq a}{L}\, \frac{f(x)}{\phi(x)} = \frac{\underset{x \doteq a}{L}\, f(x)}{\underset{x \doteq a}{L}\, \phi(x)}, \quad \underset{x \doteq a}{L}\, \phi(x) \neq 0.$$

Theorem V. *If $F(y)$ is a continuous function of y, and y is any function of x, say $y = \phi(x)$, such that $\underset{x \doteq a}{L}\, \phi(x) = b$,*

then
$$\underset{x \doteq a}{L}\, F[\phi(x)] = F[\underset{x \doteq a}{L}\, \phi(x)].$$

---

* For formal proof of these theorems see *First Course in Calculus*, or Rietz and Crathorne's *College Algebra*.

The following example will illustrate the use of this theorem:

Ex. Find the limit $\underset{x \doteq 2}{L} \log (x^2 + 2x + 1)$.

Since the logarithm is a continuous function for all values different from zero, we may apply the above theorem and write

$$\underset{x \doteq 2}{L} \log (x^2 + 2x + 1) = \log \underset{x \doteq 2}{L} (x^2 + 2x + 1) = \log 9.$$

In fact, we may always write

$$\underset{x \doteq a}{L} \log \phi(x) = \log \underset{x \doteq a}{L} \phi(x),$$

provided the limit $\underset{x \doteq a}{L} \phi(x)$ is different from zero.

THEOREM VI. *If a given function lies between the values of two other functions which approach a common limit, the given function approaches the same limit.*

Thus, if $$\phi(x) \leqq f(x) \leqq \psi(x),$$

and $$\underset{x \doteq a}{L} \phi(x) = \underset{x \doteq a}{L} \psi(x) = A;$$

then $$\underset{x \doteq a}{L} f(x) = A.$$

Ex. Show that $\underset{\theta = 0}{L} \dfrac{\sin \theta}{\theta} = 1$, and $\underset{\theta \doteq 0}{L} \dfrac{\tan \theta}{\theta} = 1$.

In Fig. 5, an arc AD is described with a radius $OA = r$, and angle $AOD = \theta$. Then we have, if θ is expressed in radians,

$$\text{arc } AD = r\theta,$$
$$AB = r \sin \theta,$$
$$AC = r \tan \theta.$$

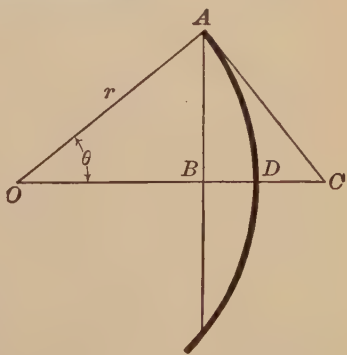

FIG. 5.

From geometry *

$$AB < AD < AC,$$

or $$r \sin \theta < r\theta < r \tan \theta.$$

Dividing by $r \sin \theta$, we get

$$1 < \frac{\theta}{\sin \theta} < \sec \theta.$$

Now $\underset{\theta \doteq 0}{L} \sec \theta = 1$; hence, since the fraction $\dfrac{\theta}{\sin \theta}$ lies between two values whose limits are equal, its limit must be the same, that is,

$$\underset{\theta \doteq 0}{L} \frac{\theta}{\sin \theta} = \underset{\theta \doteq 0}{L} \frac{\sin \theta}{\theta} = 1.$$

* See Holgate's *Geometry*, Arts. 356–361.

Dividing the members of the inequality by $r \tan\theta$, we get

$$\cos\theta < \frac{\theta}{\tan\theta} < 1.$$

Since $\underset{\theta \doteq 0}{L} \cos\theta = 1$, it follows that

$$\underset{\theta \doteq 0}{L} \frac{\theta}{\tan\theta} = 1 = \underset{\theta \doteq 0}{L} \frac{\tan\theta}{\theta}.$$

14. Limit of a monotone function. It is not always easy to find a limit of a function. Moreover, in some discussions we are not so much concerned as to what the limit is as to whether the given function *has* a limit. The following propositions, which we give without proof,* will aid us in answering this question for certain types of functions:

(a) *If a function never decreases, but always remains less than some constant, then it has a limit.*

(b) *If a function never increases but always remains greater than some constant, then it has a limit.*

Functions of the kind described in (a) and (b) belong to a class of functions called **monotone functions.**

The following illustration will perhaps aid in making clear the significance of these statements.

Given

$$\begin{aligned} f(n) &= 0.333 \cdots \\ &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \cdots. \end{aligned}$$

As n increases $f(n)$ never decreases, but always remains less than 0.4. For $n = 2$, $f(n)$ lies between 0.3 and 0.4; for $n = 3$, between 0.33 and 0.34; for $n = 4$, between 0.333 and 0.334, etc. It is evident that as n increases, the range of values within which $f(n)$ must lie constantly decreases. Consequently, there must be some constant such that we can make $f(n)$ differ from it by as little as we please by giving n an integral value sufficiently large. We know in this case that the constant in question is $\frac{1}{3}$, but that is not essential in establishing the fact that $f(n)$ has a limit as n becomes infinite.

---

* Geometrical considerations make these propositions plausible. The formal proof, while not difficult, is scarcely within the scope of this book. See Pierpont's *Theory of Functions*, Art. 108.

EXERCISES

Verify the following.

1. $\underset{x \doteq 3}{L} \dfrac{x^3 - 3x + 7}{x^2 - 5} = \dfrac{25}{4}.$

2. $\underset{h \doteq 0}{L} \dfrac{(x-h)^2 - 3hx}{x(x+h)} = 1.$

3. $\underset{x \doteq y}{L} \sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \sin y.$

4. $\underset{x \doteq 0}{L} \dfrac{\log(a+x)}{e^x} = \log a.$

5. $\underset{x \doteq 4}{L} \log\dfrac{(x^2-6)}{x+1} = \log 2.$

6. $\underset{\theta \doteq \frac{\pi}{2}}{L} \log \sin\theta = 0.$

7. $\underset{x \doteq 2}{L} [\log(x^2-1) - \log(x-1)] = \log 3.$

15. Indeterminate forms. It may happen that for some value of the variable the function takes an illusory or indeterminate form, as $\dfrac{0}{0}$. The evaluation of such an indeterminate form will be of value in finding the derivatives of functions, a process to be considered formally in the next chapter.

Consider, for example, the function $y = \dfrac{x^2-4}{x-2}$. For every value of the variable x other than $x = 2$, the function has a definite value, but for $x = 2$ it becomes $\dfrac{4-4}{2-2} = \dfrac{0}{0}$. Strictly speaking, the function $\dfrac{x^2-4}{x-2}$ has no definite value for $x = 2$; hence, in order to define the function completely for every value of the variable, we must assign it a value for this particular value of the variable. To guide us in our definition, we make use of a simple graphical representation. For x different from $x = 2$, the equation

$$y = \frac{x^2-4}{x-2}$$

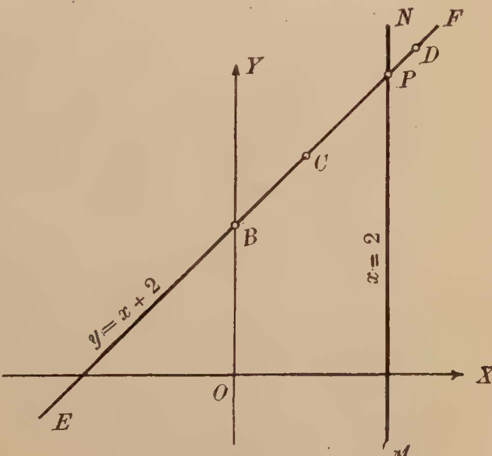

FIG. 6.

reduces to $$y = x + 2, \tag{1}$$
whose graph is the line EF, Fig. 6. Hence for values of x other than 2, the values of the function are represented graphically by points on the line EF, as B, C, D. But when $x = 2$ the function has no definite value, and hence we may represent it by any point whatever on the line $x = 2$, that is, on MN. Of the values which might be assigned to the function for $x = 2$, there is one value, represented by the point P, which is the limit of the values represented by the points on EF as x approaches 2; and it is convenient to select this value of y as the value of the function when $x = 2$.

We define therefore the value of the function for the critical value of the variable as the limit that the function approaches as the variable approaches the value for which the function becomes indeterminate. Thus for $x = 2$, the value of the function $\dfrac{x^2-4}{x-2}$ is defined as $\underset{x \doteq 2}{L} \dfrac{x^2-4}{x-2}$.

Any definition of a function for those values of the variable for which it becomes indeterminate is of course merely a convention; but the definition adopted in this case is a very useful convention, because by it the function becomes continuous for the values in question.

In general, if $f(x)$ is indeterminate for any value $x = a$, the value of the function for $x = a$ is defined by the limit $\underset{x \doteq a}{L} f(x)$. The complete definition of a function when it assumes an indeterminate form involves, therefore, the determination of a limit.

Among the indeterminate forms that a function may assume are the following:

$$\frac{0}{0}; \quad \frac{\infty}{\infty}; \quad \infty - \infty; \quad 0 \times \infty; \quad 0^0; \quad \infty^0; \quad 1^{\infty}.$$

An example of the first form is given by the function $\dfrac{\sin x}{x}$ which for $x = 0$ takes the form $\dfrac{0}{0}$. The value of this function for $x = 0$ is therefore defined by the limit

$$\underset{x \doteq 0}{L} \frac{\sin x}{x} = 1. \qquad \text{(Art. 13)}$$

The function $x \log x$ for $x = 0$ has the form $0 \times \infty$, and the function $(1+x)^{\frac{1}{x}}$ for $x=0$ has the form 1^{∞}. By proper transformations all these indeterminate forms may be reduced to the form $\frac{0}{0}$.

In many cases the limits are easily found by obvious algebraic transformations or by the use of series. In the evaluation of the indeterminate form of a rational function, that is, the quotient of two polynomials, the following principles may be used to advantage.

1. If the function is of the form $\frac{\phi(x)}{\psi(x)}$ and takes the form $\frac{0}{0}$ for $x=0$, divide both numerator and denominator by the lowest power of x that occurs in either numerator or denominator. If the fraction becomes $\frac{\infty}{\infty}$ for $x=\infty$, divide both terms of the fraction by the highest power of x in either numerator or denominator.

2. If the function has the form $\frac{\phi(x)}{\psi(x)}$ and takes the form $\frac{0}{0}$ for $x=a$, divide both $\phi(x)$ and $\psi(x)$ by the highest power of $(x-a)$ common to both.

The student should note that we do not divide the terms of the fraction by zero or infinity. For example, while the division by the factor $(x-a)$ holds only for x different from a, the values of x may, however, be taken as close as we please to a. Hence, dividing first by this factor and afterwards passing to the limit, we obtain the proper result. Indeterminate forms will be again considered in Chapter XV.

The following examples illustrate some methods that may be used in evaluating various indeterminate forms.

Ex. 1. Find the value of $\frac{x^2-6x}{x^3-4x^2-12x}$ for $x=0$.

$$\underset{x \doteq 0}{L} \frac{x^2-6x}{x^3-4x^2-12x} = \underset{x \doteq 0}{L} \frac{x-6}{x^2-4x-12} = \frac{1}{2}.$$

Ex. 2. Evaluate the function $\frac{2x^2+3x^3}{x+5x^3}$ for $x=\infty$.

$$\underset{x=\infty}{L} \frac{2x^2+3x^3}{x+5x^3} = \underset{x=\infty}{L} \frac{\frac{2}{x}+3}{\frac{1}{x^2}+5} = \frac{0+3}{0+5} = \frac{3}{5}.$$

Ex. 3. Find the value of $\dfrac{x^3-3x^2+5x-3}{2x^3-5x^2+8x-5}$ for $x=1$.

$$\frac{x^3-3x^2+5x-3}{2x^3-5x^2+8x-5}=\frac{(x^2-2x+3)(x-1)}{(2x^2-3x+5)(x-1)}=\frac{x^2-2x+3}{2x^2-3x+5}.$$

Hence, we have $\underset{x\doteq 1}{L}\dfrac{x^3-3x^2+5x-3}{2x^3-5x^2+8x-5}=\underset{x\doteq 1}{L}\dfrac{x^2-2x+3}{2x^2-3x+5}=\dfrac{1}{2}.$

Ex. 4. Evaluate $\dfrac{a-\sqrt{a^2-x^2}}{x^2}$ for $x=0$.

This function takes the form $\dfrac{0}{0}$. To find its value for $x=0$, we multiply both terms of the fraction by the complementary surd $a+\sqrt{a^2-x^2}$. We then have

$$\frac{a-\sqrt{a^2-x^2}}{x^2}=\frac{a^2-(a^2-x^2)}{x^2(a+\sqrt{a^2-x^2})}=\frac{x^2}{x^2(a+\sqrt{a^2-x^2})}=\frac{1}{a+\sqrt{a^2-x^2}}.$$

and $$\underset{x\doteq 0}{L}\frac{a-\sqrt{a^2-x^2}}{x^2}=\underset{x\doteq 0}{L}\frac{1}{a+\sqrt{a^2-x^2}}=\frac{1}{2a}.$$

Ex. 5. Evaluate $\dfrac{1-\cos\theta}{\theta}$ for $\theta=0$.

We have

$$\frac{1-\cos\theta}{\theta}=\frac{2\sin^2\frac{\theta}{2}}{\theta}=\frac{\theta}{2}\left[\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right]^2$$

$$\underset{\theta\doteq 0}{L}\frac{\theta}{2}\left[\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right]^2=\underset{\theta\doteq 0}{L}\frac{\theta}{2}\,\underset{\theta\doteq 0}{L}\left[\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right]^2=0.$$

Hence, for $\theta=0$, $\dfrac{1-\cos\theta}{\theta}$ is defined as 0.

EXERCISES

Find the limiting values of the following functions for the given values of the variables.

1. $\dfrac{4x^3-3x}{2x^3-3x^2+5x}$, $x=0$.

2. $\dfrac{x^3-3x^2-4x+12}{x^2+4x-21}$, $x=3$.

3. $\sqrt{1+x}-\sqrt{x}$, $x=\infty$.

4. $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$, $x=0$.

5. $\dfrac{x^3-64}{x-4}$, $x=4$.

6. $\dfrac{x^5-a^5}{x-a}$, $x=a$.

7. Show that $\underset{\theta \doteq 0}{L} \dfrac{\sin n\theta}{\sin \theta} = n.$

Suggestion: $\dfrac{\sin n\theta}{\sin\theta} = n\dfrac{\dfrac{\sin n\theta}{n\theta}}{\dfrac{\sin\theta}{\theta}}, \quad L\dfrac{\dfrac{\sin n\theta}{n\theta}}{\dfrac{\sin\theta}{\theta}} = \dfrac{L\dfrac{\sin n\theta}{n\theta}}{L\dfrac{\sin\theta}{\theta}} = \dfrac{1}{1}.$

8. Show that for $\theta = 0$, $\dfrac{1-\cos\theta}{\sin\theta} = 0.$

9. Show that $\underset{\theta \doteq \frac{\pi}{2}}{L} \dfrac{\sec\theta}{\tan\theta} = 1.$

10. Show that $\underset{x \doteq 0}{L} \dfrac{1}{x} \sin\dfrac{x}{2} = \dfrac{1}{2}.$

11. Evaluate $\dfrac{x^3 + 3x^2 - 5x}{3x^4 - 2x^3 + 6x}$ for $x = 0$ and $x = \infty$.

12. Evaluate $\underset{x \doteq 0}{L} \dfrac{\operatorname{arc}\tan x}{x}.$

13. Show that $\underset{\theta \doteq 0}{L} \dfrac{1-\cos\theta}{\theta^2} = \dfrac{1}{2}.$

MISCELLANEOUS EXERCISES

1. Given $y = x^2 - 3x + 5$. (*a*) Find the increment Δy corresponding to the increment Δx of the variable. (*b*) Calculate Δy for $x_1 = 3$ and $\Delta x = 0.1$.

2. Given $y = x^3$. (*a*) Find the expression for the increment Δy for $x = x_1$. (*b*) Find the limit of the ratio $\dfrac{\Delta y}{\Delta x}$ as $\Delta x \doteq 0$.

3. Given $y = \sin x$. Take $x_1 = 30°$, and assume for Δx the following values: $1°$, $30'$, $5'$. Make a table showing the values of $\sin(x_1 + \Delta x)$, Δy, and $\dfrac{\Delta y}{\Delta x}$. See if the ratio approaches a limit.

4. If $\phi(x) = a^x$, show that $[\phi(x)]^2 = \phi(2x)$.

5. If $\phi(y) = e^y + e^{-y}$, show that $\phi(3y) = [\phi(y)]^3 - 3\phi(y)$

and
$$\phi(x+y)\,\phi(x-y) = \phi(2x) + \phi(2y).$$

6. Give a physical illustration of a function that approaches a limit as the variable increases indefinitely.

7. For which of the trigonometric functions is $f(\theta) = f(-\theta)$?

8. From geometry or physics give two or more examples of monotone functions.

9. Suppose that air inclosed in a cylinder has an initial volume of 10 cubic feet and an initial pressure of 20 lbs. per square inch. Assuming that air expands according to Boyle's law, viz. $pv =$ constant, calculate the value of $\dfrac{\Delta p}{\Delta v}$ for $\Delta v = 2, 1, 0.1, 0.01$. Determine the same ratio for $\Delta v = h$, and show that as $\Delta v \doteq 0$, this ratio has the limit -2.

10. Show that $\underset{x \doteq \frac{\pi}{2}}{L} (\sec x - \tan x) = 0.$

11. Examine the function $e^{\frac{1}{x}}$ for continuity. Draw the graph of the function from $x = -2$ to $x = +2$.

12. Find the limiting values of the following functions for the given value of the variable:

(*a*) $\dfrac{\tan \theta - \sin \theta}{\theta^3}$, $\theta = 0$. (*b*) $\dfrac{\sec \theta - 1}{\theta^2}$, $\theta = 0$.

13. If n is a positive integer show that

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

14. If the variables in any given functional relation are interchanged (as $y = e^x$, $x = e^y$), show that the graphs of the two functions are symmetrical with respect to the line $x - y = 0$.

CHAPTER II

DERIVATIVES OF ALGEBRAIC FUNCTIONS

16. Increments. The idea of a variable as employed throughout the calculus is but little used in elementary mathematics. There the symbols x, y, etc., stand usually for unknown but fixed numbers whose values are to be determined. Here, as in the analytic geometry, we associate the same symbols with magnitudes that change or grow according to some law determined by the nature of the problem under discussion. As we have already seen, this idea of growth is of prime importance in the calculus, and it lies at the basis of the discussion in the present chapter.

If a variable be assigned some arbitrary value as x_1, a function of the variable takes a corresponding value $f(x_1)$; and if the variable changes to any other value x, the function takes a corresponding value $f(x)$. The changes $x - x_1$ and $f(x) - f(x_1)$ of the variable and of the function, respectively, are called **increments** of the variable and of the function and are denoted by Δx and $\Delta f(x)$. See Fig. 7. We have therefore

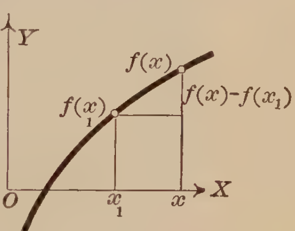

FIG. 7.

$$\Delta x = x - x_1,$$
$$\Delta f(x) = f(x) - f(x_1)$$
$$= f(x_1 + \Delta x) - f(x_1).$$

If we have $y = f(x)$, then instead of writing $\Delta f(x)$ we can equally well write Δy.

Given the increment of the variable, we may calculate the corresponding increment of the function, as shown in the following examples.

Ex. 1. Given the function

$$f(x) = 3x^2 + 4x + 2.$$

Then
$$\Delta f(x) = f(x_1 + \Delta x) - f(x_1)$$
$$= 3(x_1 + \Delta x)^2 + 4(x_1 + \Delta x) + 2 - (3x_1^2 + 4x_1 + 2)$$
$$= 6x_1\Delta x + 3(\Delta x)^2 + 4\Delta x.$$

Ex. 2. Given the law of falling bodies $s = \frac{1}{2}gt^2$. Calculate Δs corresponding to $\Delta t = \frac{1}{2}$, $t_1 = 4$.

We have here
$$\Delta s = \tfrac{1}{2}g(t_1 + \tfrac{1}{2})^2 - \tfrac{1}{2}gt_1^2$$
$$= \tfrac{1}{2}g(4 + \tfrac{1}{2})^2 - \tfrac{1}{2}g(4)^2 = 2\tfrac{1}{8}g.$$

Increments are, in general, variables and may be either positive or negative.

As was shown in the illustrative problems, Arts. 4 and 5, the ratio of the increment of the function to that of the variable is useful in giving the slope of a secant line, the mean speed of a moving point, etc. In fact, this ratio of the increments always gives a mean value of some kind. It is, however, the *limit* of this ratio, the derivative, that is of special importance.

17. Definition of a derivative. Consider the ratio

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x},$$

x_1 being some particular value of x. For $\Delta x = 0$ this ratio assumes the indeterminate form $\frac{0}{0}$, and to evaluate it for this value of Δx we must find its limit as $\Delta x \doteq 0$, according to the principles laid down for the evaluation of indeterminate forms. This limit is called the **derivative** of $f(x)$ with respect to x for the value $x = x_1$. We shall denote derivatives with respect to the variables x, t, etc., by the symbols D_x, D_t, etc. Thus we have for any value of x,

$$D_x f(x) = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x}. \qquad (1)$$

The derivative may therefore be defined *as the limit of the ratio of the increment of the function to that of the variable as the latter increment approaches the value zero.*

Instead of the symbol D_x, other symbols are often employed. For example, having

$$y = f(x),$$

we may indicate the derivative by any one of the following symbols:

$$f'(x),\ y_x',\ y',\ Dy.$$

The last two symbols are used only when there is no ambiguity as to the independent variable.

The derivative is also referred to as the **differential coefficient** or as the **derived function**,* and the process of finding the derivative is called **differentiation**. This process consists of the following steps:

1. Give to the independent variable an increment.
2. Calculate the corresponding increment of the function.
3. Find the ratio of the increment of the function to that of the variable.
4. Determine the limit of this ratio as the increment of the variable is allowed to approach zero.

The following examples illustrate the process of differentiation.

Ex. 1. Find the derivative of $y = f(x) = 5x^2 - 3x + 4$.

First, giving the variable the increment Δx, we have

$$\begin{aligned} y + \Delta y = f(x + \Delta x) &= 5(x + \Delta x)^2 - 3(x + \Delta x) + 4 \\ &= 5x^2 - 3x + 4 + (10x - 3)\Delta x + 5(\Delta x)^2. \end{aligned}$$

Subtracting, we obtain as the increment of the function

$$\Delta y = f(x + \Delta x) - f(x) = (10x - 3)\Delta x + 5(\Delta x)^2.$$

The result of dividing by Δx is the ratio

$$\frac{\Delta y}{\Delta x} = 10x - 3 + 5\Delta x;$$

and the limit of this ratio as Δx approaches zero is

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta x \doteq 0}{L} (10x - 3 + 5\Delta x) = 10x - 3.$$

Hence, $$f'(x) = D_x(5x^2 - 3x + 4) = 10x - 3.$$

To find the value of the derivative for some particular value of x, we merely substitute that value of x in the derived function $f'(x)$. Thus, for $x = x_1$, $f'(x_1) = 10x_1 - 3$; for $x = 5$, $f'(5) = 10 \times 5 - 3 = 47$.

---

* The word "derivative" is frequently used to mean the value of the limit (1) for a particular value of x. The term "derived function" refers more properly to the assemblage of all such values.

Ex. 2. Given the equation expressing Boyle's law, viz. $pv = k$, or $p = \frac{k}{v}$. Find the derivative of p with respect to v.

We have
$$p = \frac{k}{v},$$
whence
$$p + \Delta p = \frac{k}{v + \Delta v}.$$
Subtracting, we get for the increment of p,
$$\Delta p = \frac{k}{v + \Delta v} - \frac{k}{v} = -k\frac{\Delta v}{v(v + \Delta v)},$$
whence
$$\frac{\Delta p}{\Delta v} = -k\frac{1}{v(v + \Delta v)}.$$
and
$$\underset{\Delta v \doteq 0}{L}\frac{\Delta p}{\Delta v} = -k \underset{\Delta v \doteq 0}{L}\frac{1}{v(v + \Delta v)} = -\frac{k}{v^2}.$$
Hence
$$D_v p = f'(v) = -\frac{k}{v^2}.$$
For $k = 200$, and $v = 10$ (see Ex. 9, p. 25),
$$f'(10) = -\frac{200}{10^2} = -2.$$

EXERCISES

Find the derivatives of the following functions, using the general process described in this section.

1. $y = x^4$.
2. $y = x^2 - 4x + 5$.
3. $y = \frac{a}{x^2}$.
4. $y = x^{\frac{1}{2}}$.
5. $y = x^3 - x + 2$.
6. $y = \frac{x}{x - 1}$.
7. $y = (x - a)^3$.
8. $s = \frac{1}{2}gt^2$.
9. $s = at + \frac{1}{2}gt^2$.
10. $\rho = a\theta + b\theta^{-1}$.
11. If $p(v - b) = k$, find $D_v p$.

18. Conditions for a derivative. Not every function has a derivative for all values of the variable. In order that the ratio
$$\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{1}$$
shall have a limit as $\Delta x \doteq 0$, it is necessary that the numerator shall approach zero simultaneously with the denominator. In other words, we must have
$$\underset{\Delta x \doteq 0}{L} f(x_1 + \Delta x) = f(x_1),$$

which is equivalent to writing

$$\underset{x \doteq x_1}{L} f(x) = f(x_1).$$

This is, however, the condition for the continuity of $f(x)$ for the value $x = x_1$. It follows that a function cannot have a derivative at a point of discontinuity.

Although the numerator of (1) must vanish simultaneously with the denominator, it may happen that as $\Delta x \doteq 0$ the ratio becomes infinite for particular values of the variable. In such cases, we say that for the particular value of x in question, say $x = x_1$, the value of the derivative becomes infinite, and write

$$f'(x_1) = \infty.$$

19. Geometrical and physical significance of the derivative. It has been pointed out that the ratio $\dfrac{\Delta f(x)}{\Delta x}$ gives the slope of the secant line passing through the points whose ordinates are $y_1 = f(x_1)$, $y = f(x_1 + \Delta x)$. If Δx be allowed to approach zero, the secant line approaches as a limit the tangent to the curve $y = f(x)$ at the point (x_1, y_1).

Hence, if $y = f(x)$ is represented by a continuous curve, we have

$$f'(x_1) = \tan \phi,$$

where ϕ is the angle made with the axis of X by the tangent to the given curve at the point (x_1, y_1). For the sake of definiteness we shall take ϕ as the acute angle made with the positive direction of the X-axis. It may be either positive or negative. Tan ϕ is called the **slope** or **gradient** of the tangent to the curve; hence *the derivative of the function $f(x)$ for the value $x = x_1$ represents the slope of the curve $y = f(x)$ at the point (x_1, y_1).*

Again, it has been shown that $\dfrac{\Delta s}{\Delta t}$ gives the mean speed of the moving point during the time Δt. If Δt approaches zero, the limit, that is the value of the derivative $D_t s$ for $t = t_1$, gives the speed of the moving particle at that instant. Similarly, if m denotes the mass and V the volume of a body, the limit of the ratio $\dfrac{\Delta m}{\Delta V}$ as ΔV approaches zero gives the density at a point; and

if ΔQ denotes the heat entering a body and $\Delta\tau$ the corresponding rise in temperature, the limit of the ratio $\frac{\Delta Q}{\Delta\tau}$, as $\Delta\tau$ approaches zero, gives the specific heat at the definite temperature τ_1.

The ratio of the increments gives always a *mean* value for an interval; the derivative an *instantaneous* value or the value at a point.

20. General theorems on differentiation. While the process of finding a derivative described in Art. 16 is sufficient for the differentiation of all functions that can be differentiated, it becomes inconvenient when the function is complicated, and the labor of differentiation can be abridged by the use of certain general theorems that apply to all classes of functions. These theorems are given in the following Arts. 21–29. Because of their fundamental importance, the student should have them at his ready command.

21. Derivative of a constant.

Let $$y = c.$$

Then $$y + \Delta y = c.$$

Subtracting, we have $$\Delta y = 0,$$

whence $$D_x y = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = 0,$$

or $$\boldsymbol{D_x c = 0}.$$

We have therefore the following theorem:

Theorem I. *The derivative of a constant is zero.*

Geometrically the equation $y = c$ is represented by a straight line parallel to the X-axis. The slope of this line is zero in accordance with Theorem I.

22. Derivative of a variable with respect to itself.

Given $$y = x,$$

then $$y + \Delta y = x + \Delta x,$$

and $$\Delta y = \Delta x.$$

Hence $$D_x y = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta x \doteq 0}{L} \frac{\Delta x}{\Delta x} = 1;$$

that is, $$\boldsymbol{D_x x = 1}.$$

This result gives the following theorem:

THEOREM II. *The derivative of a variable with respect to itself is unity.*

Ex. Give a geometrical illustration of Theorem II.

23. Derivative of a sum.

Let $$y = f(x) + \phi(x).$$

Then $$y + \Delta y = f(x + \Delta x) + \phi(x + \Delta x),$$

and $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}.$$

Taking the limits of both members, we have

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \underset{\Delta x \doteq 0}{L} \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x},$$

or $$D_x y = D_x f + D_x \phi.$$

This process may be extended to the sum of any finite number of functions. Thus, if we have

$$y = u + v + w,$$

where u, v, w are functions of x, we may write

$$\boldsymbol{D_x y = D_x(u + v + w) = D_x u + D_x v + D_x w}.$$

We may, therefore, state the following theorem:

THEOREM III. *The derivative of the algebraic sum of a finite number of terms is the algebraic sum of their derivatives.*

24. Derivative of a function plus a constant.

Let $$y = f(x) + c.$$

By the application of Theorem III, we have

$$\begin{aligned} D_x y &= D_x f(x) + D_x c \\ &= D_x f(x). \qquad \text{(Art. 21)} \end{aligned}$$

That is, $$D_x[f(x) + c] = D_x f(x).$$

If u be used to denote $f(x)$, this result may be written

$$\boldsymbol{D_x(u+c) = D_x u}.$$

We have, therefore, as a corollary to Theorem III, the following:

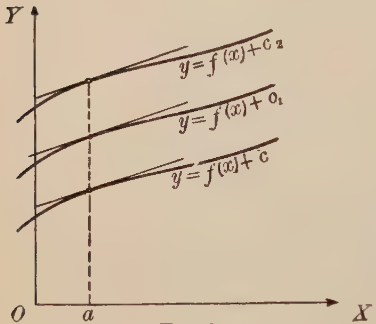

FIG. 8.

COROLLARY. *The derivative of a function is not affected by increasing or decreasing the function by an additive constant.*

It follows also that two functions differing only by a constant term have the same derivative. In the process of differentiating, the constant terms may consequently be neglected.

Geometrically, this corollary has an interesting significance. Suppose the function $y = f(x) + c$ to represent some curve. The effect of adding or subtracting a constant term, that is, of changing the value of c, is simply to shift the curve up or down with reference to the X-axis. (See Fig. 8.) As has been shown, the derivative measures the slope of the curve, and it is evident that this slope for any particular value of x, as a, is not changed by shifting the curve as indicated.

This corollary has also an important significance in the inverse operation of integration, as we shall see later.

25. Derivative of a product.

Given $$y = f(x) \cdot \phi(x).$$

We have then $$y + \Delta y = f(x + \Delta x) \cdot \phi(x + \Delta x),$$

and $$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) \cdot \phi(x+\Delta x) - f(x) \cdot \phi(x)}{\Delta x}.$$

By adding and subtracting $f(x) \cdot \phi(x + \Delta x)$ in the numerator, this ratio may be written in the form

$$\frac{\Delta y}{\Delta x} = \phi(x+\Delta x) \cdot \frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x) \cdot \frac{\phi(x+\Delta x) - \phi(x)}{\Delta x}.$$

Since $\phi(x)$ is by hypothesis continuous, $\phi(x+\Delta x)$ in the limit becomes $\phi(x)$; hence, we have upon passing to the limit,

$$D_x y = \phi \cdot D_x f + f \cdot D_x \phi.$$

This result can be extended to the product of a finite number of functions; thus, if

$$y = uvw,$$

where u, v, w are functions of x, we have

$$\boldsymbol{D_x y = D_x (uvw) = uv \cdot D_x w + uw \cdot D_x v + vw \cdot D_x u.}$$

We have then the following theorem:

THEOREM IV. *The derivative of the product of a finite number of factors is the sum of the products obtained by multiplying the derivative of each factor by the product of all the other factors.*

26. Derivative of a constant times a function.

If we have given $\qquad y = c \cdot f(x),$

we have, from Theorem IV,

$$\begin{aligned} D_x y &= c \cdot D_x f(x) + f(x) \cdot D_x c \\ &= c \cdot D_x f(x). \end{aligned}$$

That is, $\qquad D_x c f(x) = c \cdot D_x f(x).$

Again, using u to denote $f(x)$, this result may be written

$$\boldsymbol{D_x (cu) = cD_x u.}$$

Hence we have the following corollary to Theorem IV:

COROLLARY. *The derivative of a constant times a function is the constant times the derivative of the function.*

We may now combine the results of Theorem I, the corollary to Theorem III, and the above corollary in the one general statement that in the process of differentiation *constant terms disappear but constant factors remain.*

Ex. Find the derivative of $y = x^3 + 7x + 8$.

We have

$$\begin{aligned} D_x y &= D_x(x^3) + D_x(7x) + D_x 8. && \text{(Th. III)} \\ D_x(x^3) &= D_x(xxx) = x^2 D_x x + x^2 D_x x + x^2 D_x x && \text{(Th. IV)} \\ &= 3x^2. && \text{(Th. II)} \\ D_x 7x &= 7 D_x x && \text{(Cor.)} \\ &= 7. && \text{(Th. II)} \\ D_x 8 &= 0. && \text{(Th. I)} \end{aligned}$$

Combining these results, we have

$$D_x y = 3x^2 + 7.$$

The student need not write down each step as has been done in this illustrative example; but he should make himself so familiar with the principles set forth in the preceding theorems that he can write down the results at once.

EXERCISES

Find the derivatives of the following functions.

1. $y = x^2$. 2. $y = x^2 - 10x + 4$. 3. $y = x^2(x-1)$.

4. $y = 5x^3 - x(x+2)$. 5. $y = x^2(x^2-2) + x(x+3)$.

6. Plot the curve $y = 4x^2 + c$, giving c the values 2, 4, 6, 8, and show that each of these curves has the same slope for $x = 2$.

7. Let the distance traversed by a moving point be given as a function of the time by the equation $s = 80t - 16t^2$. Derive a general expression for the speed $D_t s$, and find the speed at the end of 3 seconds.

27. Derivative of a quotient.

Let
$$y = \frac{f(x)}{\phi(x)};$$
then
$$y + \Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)},$$
and consequently
$$\Delta y = \frac{f(x+\Delta x)}{\phi(x+\Delta x)} - \frac{f(x)}{\phi(x)} = \frac{\phi(x)\cdot f(x+\Delta x) - f(x)\cdot\phi(x+\Delta x)}{\phi(x+\Delta x)\cdot\phi(x)}.$$

Subtracting and adding $\phi(x)\cdot f(x)$ in the numerator, we have, after dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\phi(x)\cdot\dfrac{f(x+\Delta x)-f(x)}{\Delta x} - f(x)\cdot\dfrac{\phi(x+\Delta x)-\phi(x)}{\Delta x}}{\phi(x+\Delta x)\cdot\phi(x)}.$$

Remembering that $\phi(x)$ is continuous and hence, as $\Delta x \doteq 0$, $\phi(x+\Delta x)$ becomes $\phi(x)$, we have for the limiting value

$$D_x y = \frac{\phi(x)\cdot D_x f(x) - f(x)\cdot D_x\phi(x)}{[\phi(x)]^2}.$$

Denoting the two functions by u and v, respectively, this result can be more conveniently written as follows:

$$D_x\left(\frac{u}{v}\right)=\frac{v\,D_xu-u\,D_xv}{v^2}.$$

We may state this result in the following theorem:

THEOREM V. *The derivative of a fraction is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

28. Derivative of the quotient of a constant by a variable.

If the numerator of the fraction is a constant, the result just obtained becomes

$$D_x\left(\frac{c}{v}\right)=-\frac{c\,D_xv}{v^2}.$$

Hence we have the following corollary to Theorem V:

COROLLARY. *The derivative of the quotient of a constant by a variable is equal to minus the product of the constant and the derivative of the variable divided by the square of the variable.*

Ex. Given $f(x)=\dfrac{x+2}{x^2+3x+1}$; find $f'(x)$.

From Theorem V, we have

$$f'(x)=\frac{(x^2+3x+1)\,D_x(x+2)-(x+2)\,D_x(x^2+3x+1)}{(x^2+3x+1)^2}$$

$$=\frac{(x^2+3x+1)\,1-(x+2)(2x+3)}{(x^2+3x+1)^2}=-\frac{(x^2+4x+5)}{(x^2+3x+1)^2}.$$

EXERCISES

Find the derivatives of the following functions:

1. $y=3x^2-4x-6.$
2. $y=4x^3-2x^2+5.$
3. $y=x^2(x-5).$
4. $y=x(x+1)(x-2).$
5. $y=x^3(x^2-2)(x+1).$
6. $y=x^2(x-4)+x(x^2+3).$
7. $y=\dfrac{x-3}{x^2}.$
8. $y=\dfrac{1}{x^2}.$
9. $y=\dfrac{m}{ax+b}.$
10. $y=\dfrac{ax+b}{cx+d}.$

11. $\rho = \theta - \dfrac{1}{\theta^2}$.

12. $s = \dfrac{5t}{t^2 - 1}$.

13. $y = \dfrac{x^2 - 4x + 5}{x^2 + 2}$.

14. $y = \dfrac{x - 6}{x^2 + 4x - 5}$.

15. Plot the curve $y = \dfrac{x^2}{4} + c$, giving c the values 1, 3, 5, and show that each of the curves has the same slope for $x = 3$.

16. Show that if the nth power of a variable occurs as a factor in the denominator of a fraction, the $n + 1$st power of the factor will occur in the denominator of the derivative of the fraction after reduction to lowest terms.

29. Derivative of u^n. Let u denote any function of x. To determine the derivative of u^n with respect to x, we proceed in the usual manner. Thus, let

$$y = u^n,$$

then

$$y + \Delta y = (u + \Delta u)^n,$$

and provided $\Delta u \neq 0$, we have

$$\frac{\Delta y}{\Delta x} = \frac{(u + \Delta u)^n - u^n}{\Delta u} \cdot \frac{\Delta u}{\Delta x}. \tag{1}$$

Since u is continuous and therefore Δu approaches zero with Δx, we have upon passing to the limit

$$D_x y = \underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta u \doteq 0}{L} \frac{(u + \Delta u)^n - u^n}{\Delta u} \cdot \underset{\Delta x \doteq 0}{L} \frac{\Delta u}{\Delta x}. \tag{2}$$

The evaluation of the first limit in the second member requires consideration of the special limit

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a}.$$

Let n be any positive integer; then by division we obtain

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1},$$

whence

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a} = \underset{x \doteq a}{L} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

In the parenthesis there are n terms, each of which has the limit a^{n-1} as $x \doteq a$. Since n is finite, the limit of the sum is equal to the sum of the limits, and we have therefore

$$\underset{x \doteq a}{L} \frac{x^n - a^n}{x - a} = na^{n-1}. \tag{3}$$

It is easily shown that (3) holds when n is a positive fraction, also when n is negative, either integral or fractional. For the first case let $n = \frac{p}{q}$ and assume $x = z^q$, $a = b^q$; for the second, put $n = -m$. The proof is similar to that just given.

In (3) let x be replaced by $u + \Delta u$ and a by u; then as $x \doteq a$, $\Delta u \doteq 0$, and (3) becomes
$$\underset{\Delta u \doteq 0}{L} \frac{(u+\Delta u)^n - u^n}{\Delta u} = nu^{n-1}. \qquad (4)$$

Using this result in (2), we obtain when n is a positive integer
$$\boldsymbol{D_x(u^n) = nu^{n-1}D_x u.} \qquad (5)$$

In Art. 65 it will be shown that formula (5) holds good when n is any real constant, rational or irrational.

If $u = x$, we have, since $D_x u = 1$, the important special case
$$\boldsymbol{D_x(x^n) = nx^{n-1}.} \qquad (6)$$

Ex. 1. Let $y = a(3x^2+7)^3$; find $D_x y$.

Put $$u = 3x^2 + 7.$$

We have then $$D_x y = a \cdot D_x u^3 = 3au^2 D_x u.$$

But $$D_x u = D_x(3x^2+7) = 6x.$$

Combining these results, we obtain
$$D_x y = 18ax(3x^2+7)^2.$$

Ex. 2. Let $y = \sqrt{x^2 - a^2}$.

Substituting, $$u = x^2 - a^2,$$

we have $$y = u^{\frac{1}{2}}.$$

Hence $$D_x y = \tfrac{1}{2}u^{-\frac{1}{2}} D_x u = \tfrac{1}{2}u^{-\frac{1}{2}} 2x = \frac{x}{\sqrt{x^2-a^2}}.$$

30. Explicit and Implicit Functions. Algebraic functions. If a function is expressed directly in terms of the variable, it is called an **explicit function** of that variable; if, on the other hand, it is not expressed directly in terms of the variable, it is called an **implicit function**. Thus,
$$y = 3x^2 + 7x + 9$$
is an explicit function of x, while in
$$3xy + 7y - 9x^2 = 4$$

y is an implicit function of x. We can express y as an explicit function of x by solving the above equation for y and thus obtain

$$y=\frac{4+9x^2}{3x+7}.$$

A function of the general form

$$y=a_0x^n+a_1x^{n-1}+\cdots+a_n$$

belongs to a class of functions known as **algebraic functions.*** In the form in which the function is here written it is also an explicit function of x. The general laws for differentiation that have been developed in the preceding articles are sufficient to enable the student to write at once the derivative of any explicit algebraic function. The differentiation of implicit functions will be considered later.

Ex. Differentiate the function $y=x^3-4x+2x^{-1}$.

From Art. 23, $\qquad D_xy=D_x(x^3)-D_x(4x)+D_x(2x^{-1})$,

whence, using (6), Art. 29, we have

$$D_xy=3x^2-4-2x^{-2}.$$

EXERCISES

Differentiate the following algebraic functions.

1. $y=x^{\frac{3}{2}}$.

2. $y=x^2-x^{-1}+5$.

3. $y=x^{\frac{1}{2}}-x^{-\frac{1}{2}}$.

4. $y=(2x-5)^4$.

5. $y=(x^2-4x+3)^2$.

6. $y=(x^2-a^2)^{\frac{3}{2}}$.

7. $y=\sqrt[3]{1+x^2}$.

8. $y=(4x+3)^{-\frac{1}{3}}$.

9. $y=\sqrt{\dfrac{x^2-a^2}{x^2+a^2}}$.

10. $y=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x^3}$.

11. $p=A+\dfrac{B}{t}+\dfrac{C}{t^2}$.

12. $\rho=\theta^2+k\theta^{\frac{1}{2}}-2$.

13. $s=80-16t^2$.

14. $y=x^p(1-x)^q$.

15. $y=x^3\sqrt{x^2-5}$.

16. $y=x^{\frac{2}{3}}(x^2-a^2)^{-\frac{1}{3}}$.

17. By differentiating $\dfrac{1}{x^n}$, show that $D_xx^n=nx^{n-1}$ holds for negative integral values of n.

18. Given $xe^{-\frac{y}{3}}=c$; express y as an explicit function of x.

---

* For a more general definition of an algebraic function, see *First Course*, p. 9.

19. Given
$$x = f(\phi) = a(\phi - \sin\phi)$$
$$y = F(\phi) = a(1 - \cos\phi),$$
express x as an explicit function of y.

31. Derivative of a function of a function. The derivative of u^n discussed in Art. 29, where u is a function of x, is a special form of the more general case which we shall now consider. Let us suppose that $y = f(u)$, and $u = \phi(x)$, where these functions have respectively the derivatives $f'(u)$ and $\phi'(x)$ for the corresponding values of the variables u and x. If we wish to find $D_x y$, we can do so by substituting the given value of u in $f(u)$, thus expressing y directly as a function of x; from this expression we can obtain the desired result by methods already explained. In many cases, however, the derivative can be obtained more easily by the following method.

Let x take an increment; then u and y will also take increments, and since u and y are continuous functions of x, these increments will approach zero as the increment of x approaches zero. We have then
$$\Delta y = f(u + \Delta u) - f(u).$$
$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \cdot \frac{\Delta u}{\Delta x}, \quad \Delta u \neq 0.$$

Since $\Delta u \doteq 0$ as $\Delta x \doteq 0$, we have in the limit
$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = \underset{\Delta u \doteq 0}{L} \frac{f(u + \Delta u) - f(u)}{\Delta u} \underset{\Delta x \doteq 0}{L} \frac{\Delta u}{\Delta x};$$
that is,
$$\boldsymbol{D_x y = D_u y \cdot D_x u.} \tag{1}$$

This formula expressed in words gives us the following theorem:

THEOREM. *If $y = f(u)$ and $u = \phi(x)$, the derivative of y with respect to x is the product of the derivative of y with respect to u and the derivative of u with respect to x.*

This theorem asserts the principle that if u changes m times as rapidly as x, and y changes n times as rapidly as u, then y changes mn times as rapidly as x. For example, if a horse travels twice as fast as a man, and a train four times as fast as the horse, the train travels eight times as fast as the man.

Ex. 1. Find $D_x y$ when $y = \sqrt{u}$ and $u = 3x^2 - 4$.

We have $$D_u y = \tfrac{1}{2} u^{-\frac{1}{2}}, \text{ and } D_x u = 6x;$$

hence $$D_x y = \tfrac{1}{2} u^{-\frac{1}{2}} \cdot 6x = 3xu^{-\frac{1}{2}} = \frac{3x}{\sqrt{3x^2 - 4}}.$$

Ex. 2. Find $D_x y$ when $y = (a^2 - x^2)^{-\frac{3}{2}}$.

Put $u = a^2 - x^2$; then $y = u^{-\frac{3}{2}}$, $D_u y = -\frac{3}{2} u^{-\frac{5}{2}}$, and $D_x u = -2x$.

Therefore $$D_x y = 3xu^{-\frac{5}{2}} = \frac{3x}{(a^2 - x^2)^{\frac{5}{2}}}.$$

EXERCISES

Differentiate the following functions:

1. $y = \sqrt[3]{x^2 + 5}$.
2. $y = (x^2 - 2x + 5)^{\frac{1}{2}}$.
3. $y = (a^2 - x^2)^{\frac{5}{2}}$.
4. $y = (a^2 - x^2)^{-\frac{5}{2}}$.
5. $y = (x^{\frac{1}{3}} + a^{\frac{1}{3}})^2$.
6. $y = \left(\dfrac{x^4}{x^2 - a^2}\right)^{\frac{1}{3}}$.
7. $y = \sqrt[3]{3x^2 - 4ax}$.
8. $y = \sqrt{(x + m)(x + n)}$.
9. $y = (x^3 - 2x + 5)^{-\frac{1}{3}}$.
10. $y = 3(x^2 - 4)^{\frac{1}{2}}(x + 3)^{\frac{3}{2}}$.
11. $\rho = \sqrt{\dfrac{\theta}{1 + \theta^2}}$.
12. $\rho = (1 + \theta^2)^{-\frac{1}{2}}$.
13. $y = (x + \sqrt{x^2 - a^2})^{\frac{1}{2}}$.
14. $y = \dfrac{x^2 + 2}{\sqrt{x^2 + 1}}$.
15. $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$.

16. By putting $f(u) = cu$, prove the corollary to Theorem IV, Art. 26.

17. By putting $f(u) = u^n$, develop the law for differentiation of u^n given in Art. 29.

18. Write a function which has x^2 as its derivative.

19. Show that the curve for which the slope of the tangent line (that is, $\tan \phi$) is numerically equal to the abscissa has the general form $y = \frac{1}{2}x^2 + C$.

20. Find the values of x for which the derivative of $f(x) = x^3 - 9x^2 + 24x$ becomes zero. What is the geometrical interpretation?

21. At what points of the curve $y = x^3 - 12x$ is the tangent parallel to the X-axis?

32. Derivatives of inverse functions. If y is given as an explicit function of x, and it is possible to solve for x in terms of y, we may then express x as an explicit function of y. Thus, if

$$y = x^2 - 4,$$

we have
$$x = \pm\sqrt{y+4}.$$

In general, let $x = \phi(y)$ be the result of solving the equation $y = f(x)$ for x; then $f(x)$ and $\phi(y)$ are said to be **inverse functions.** Examples of inverse functions are $\sin x$ and $\text{arc}\sin y$, $\log x$ and e^y, x^n and $y^{\frac{1}{n}}$, etc.

In the process of differentiation we ordinarily express y as an explicit function of x and determine D_xy directly. Sometimes, however, it is convenient to express x as a function of y and find D_xy by means of the inverse function. To find the relation between the two derivatives, we may proceed as follows:

Given
$$y = f(x); \tag{1}$$
and let the inverse function be
$$x = \phi(y). \tag{2}$$

By differentiation, we have from (2) by aid of Art. 31,

$$1 = D_y\phi(y) \cdot D_xy,$$

or
$$1 = D_yx \cdot D_xy.$$

Hence we have, provided $D_yx \neq 0$,

$$D_xy = \frac{1}{D_yx}. \tag{3}$$

This formula states the principle that if y is changing n times as rapidly as x, then x is changing $\frac{1}{n}$th as rapidly as y.

Ex. Given the equation of the parabola $y^2 = 4px$; find D_xy. Solving for x, we have

$$x = \frac{y^2}{4p},$$

whence
$$D_yx = \frac{2y}{4p} = \frac{y}{2p}.$$

From the theorem of inverse functions, we have

$$D_xy = \frac{1}{D_yx} = \frac{2p}{y} = \frac{2p}{\sqrt{4px}} = \sqrt{\frac{p}{x}}.$$

EXERCISES

1. Find $D_x y$ when $y^3 = x - 4$.

2. Find $D_x y$ when $\frac{y-2}{y+1} = x^2$.

3. If $\rho^2 = a\theta + b$, find $D_\theta \rho$.

4. If $v^2 = 2\,gs$, find $D_s v$.

5. Find functions inverse to the following functions:

$$(a)\ y = x^3;\quad (b)\ y = \log(x^2 + a^2);\quad (c)\ y = \sqrt{a^2 - x^2}.$$

6. Find the function inverse to

$$f(x) = \log(x + \sqrt{x^2 + 1}).$$

33. Derivative of one function with respect to another, when both are functions of a common variable.

Suppose we have given

$$y = f(t), \qquad x = \phi(t),$$

and that we wish to find $D_x y$. We have, for $\Delta t \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}}.$$

Because of the continuity of $f(t)$ and $\phi(t)$, $\Delta y \doteq 0$ and $\Delta x \doteq 0$ simultaneously as $\Delta t \doteq 0$. Hence, by Art. 17, we have upon passing to the limit.

$$D_x y = \frac{D_t y}{D_t x}.$$

Ex. **1.** Given $y = z^2 + 1$, and $x = \sqrt{z}$; find $D_x y$.

We have $$D_z y = 2\,z,$$

and $$D_z x = \tfrac{1}{2} z^{-\frac{1}{2}}.$$

Hence, $$D_x y = \frac{2\,z}{\frac{1}{2} z^{-\frac{1}{2}}} = 4\,z^{\frac{3}{2}} = 4\,x^3.$$

Ex. **2.** The equations $s = \frac{1}{2} gt^2$ and $v = gt$ refer to falling bodies. Find the derivative $D_s v$.

$$D_s v = \frac{D_t v}{D_t s} = \frac{g}{gt} = \frac{1}{t} = \frac{g}{v}.$$

EXERCISES

1. Find $D_x y$, when $y = 3\,t^2 - t - 10$ and $x = t + 8$.

2. Find $D_x y$, when $y = t^4 - 3\,t^2 + 7$ and $x = t^2 - 2\,t + 4$. Find the values of this derivative for $t = 0, 2, 5$.

3. If $\rho = \sqrt{t}$ and $\theta = t^2 - 10$, find $D_\theta \rho$.

4. If $x = at$ and $y = bt - \frac{1}{2} ct^2$, find $D_x y$ and $D_y x$.

5. Work examples 3 and 4 by eliminating t before differentiating.

6. Prove the theorem of Art. 33 by making use of the theorems of Arts. 31 and 32.

MISCELLANEOUS EXERCISES

Differentiate the following functions with respect to the variable indicated.

1. $y = 5x^4 + 3x^3 - 8x + 7.$

2. $y = x^2(3x^2 - 4)(x+1).$

3. $y = x^{\frac{1}{3}}(x^{\frac{1}{2}} - a^{\frac{1}{2}})^2.$

4. $y = \dfrac{9 - 4x}{2(x-2)^2}.$

5. $y = \dfrac{(x^2 - 2)}{3}\sqrt{1 + x^2}.$

6. $y = \dfrac{(1 + 2x^2)\sqrt{x^2 - 1}}{x^3}.$

7. $y = \frac{1}{3}(\sqrt{x} + a)^{\frac{3}{2}} - a(\sqrt{x} + a)^{\frac{1}{2}}.$

8. $y = \dfrac{x^n}{(1+x)^n}.$

9. $y = \dfrac{1}{x^n(a+x)^m}.$

10. $y = (x + \sqrt{x^2 - 1})^n.$

11. $y = \sqrt{x + \sqrt{1 + x^2}}.$

12. $y = \sqrt{\dfrac{x-1}{x+1}}.$

13. $y = \sqrt{\dfrac{x^2+5}{x-3}}.$

14. $y = \dfrac{(1-x)\sqrt{1-x^2}}{1+x}.$

15. $y = \dfrac{\sqrt{x^2+1} + \sqrt{x^2-1}}{\sqrt{x^2+1} - \sqrt{x^2-1}}.$

16. $y = \dfrac{x^3(x^2 - a^2)^{\frac{1}{3}}}{x + a}.$

17. $y = \dfrac{x}{x + \sqrt{x^2 - 1}}.$

18. $y = \dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x}.$

19. $p = \dfrac{c}{v - b} - \dfrac{a}{v^2}.$

20. Given $v = \dfrac{C}{p} - m(1 + ap)$, where m, C, and a are constants, find $D_v p$.

21. Show that the slope of the tangent to the curve $y = x^3 + 4$ is never negative. Find the slope for $x = 0$, $x = 2$. For what values of x does the slope decrease as x increases?

22. Find the angle which the tangent to the parabola $y^2 = 9x$ at the point (4, 6) makes with the X-axis.

23. Given $y = \dfrac{u+5}{\sqrt{u^2 - 15}}$ and $u = x^2 - 5$; find $D_x y$.

24. Given $s = bt + \frac{1}{2}at^2$ and $v = at$, find $D_s v$ in terms of v.

25. The equation $pv = C$ expresses Boyle's law, C being a constant. Find $D_v p$ and $D_p v$.

26. From Regnault's experiments, the heat q required to raise the temperature of a unit weight of water from 0° C. to a temperature τ is given by the equation

$$q = \tau + 0.00002\,\tau^2 + 0.0000003\,\tau^3.$$

(*a*) Find $D_\tau q$. (*b*) Calculate the numerical value of $D_\tau q$ for $\tau = 35°$.

27. The efficiency of a screw as a mechanical device is given by the relation $E = \dfrac{x(1-\mu x)}{x+\mu}$, where μ denotes the coefficient of friction and x the tangent of the pitch angle of the screw. Find the derivative $D_x E$, and its values for $x = 0$ and $x = \mu$. Find also the value of x for which $D_x E = 0$.

28. Find the slope of the tangent to the circle $x^2 + y^2 = 25$ for the points $x = 0$, $x = 4$, $x = -3$.

29. Find $D_s v$, where $v^2 = 2c\left(\dfrac{1}{s} - \dfrac{1}{a}\right)$.

30. $D_x y = \dfrac{x^2}{ax+b}$ and $u = \dfrac{a}{2}x^2 + bx$. Find $D_u y$.

31. Interpret geometrically the corollary of Art. 26.

32. Find functions which have the following derivatives:

(*a*) $3x^2$; (*b*) $4x^3 - 1$; (*c*) $x^2 - x + 2$; (*d*) $-\dfrac{1}{x^2}$.

33. Deduce Theorem V, Art. 27, from Theorem IV by means of the substitution $\dfrac{u}{v} = w$, whence $D_x u = D_x(vw)$.

34. If the factor $(x - a)$ occurs n times in a function $f(x)$, show that it occurs $n - 1$ times in the derivative $D_x f(x)$, and from this principle deduce a method of finding whether an algebraic equation has multiple roots.

CHAPTER III

ELEMENTARY APPLICATIONS OF DERIVATIVES

34. Slope of a curve. The equations of the curves thus far discussed have all been given in the form

$$y=f(x), \text{ or } F(x, y)=0. \tag{1}$$

In some cases it is more convenient to express both x and y in terms of a third variable; thus

$$x=\phi(t),\ y=\psi(t), \tag{2}$$

where $\phi(t)$, $\psi(t)$ are single-valued functions of the variable t. We call these equations the **parametric equations** of the curve. As examples of parametric equations of curves, we have for the circle the equations

$$y=a \sin \theta,\ x=a \cos \theta,$$

and for the cycloid

$$y=a(1-\cos \theta),$$
$$x=a(\theta-\sin \theta).$$

We may pass from the parametric form of expression to that given by equation (1) by eliminating the common variable.

In whichever form the equation of the curve is given, we have

$$D_x y=\tan \phi, \tag{3}$$

where ϕ denotes the acute angle between the tangent to the curve and the positive direction of the X-axis. To determine $\tan \phi$ from the parametric equations of the curve, we have by Art. 33

$$D_x y=\frac{D_t y}{D_t x}=\tan \phi. \tag{4}$$

Tan ϕ has been defined as the slope or gradient of the tangent to the curve. It may also be called the **slope of the curve**, for the direction of the tangent to the curve is also the direction of the

curve at the point of tangency. *The slope of the curve, therefore, is given by the value of the derivative $D_x y$.*

The value of the derivative at any point gives still other properties of the curve. Since the direction of the tangent is identical with the direction of the curve at the point of tangency, it follows that so long as ϕ is a positive angle the ordinates of the curve are increasing as the abscissa increases. When, however, ϕ is positive, $\tan\phi$, and hence $D_x y$, is positive; similarly, when ϕ is negative, the ordinates of the curve are decreasing as the abscissa increases. But in this case $\tan\phi$, and hence $D_x y$, is negative. If the value of $D_x y$ becomes zero for any value of x, then $\tan\phi$ is zero and ϕ must be zero; that is, the tangent to the curve at this point is parallel to the X-axis. This occurs whenever the curve has a turning point, that is, a point where the ordinates cease to increase and begin to decrease, or *vice versa*. Turning points are shown at A, C, and E, Fig. 10. If $D_x y$ becomes infinite for a particular value of x, the value of ϕ is then $\frac{1}{2}\pi$ or $-\frac{1}{2}\pi$; that is, the tangent to the curve at that point is perpendicular to the X-axis. We may summarize these results as follows:

At any point of the curve $y = f(x)$, the ordinate increases or decreases with increasing x according as $D_x y$ is positive or negative. If $D_x y$ is zero for any value of x, then at that point the tangent to the curve is parallel to the axis of abscissas. If $D_x y$ becomes infinite for any value of x, the tangent to the curve is perpendicular to the axis of abscissas at that point.

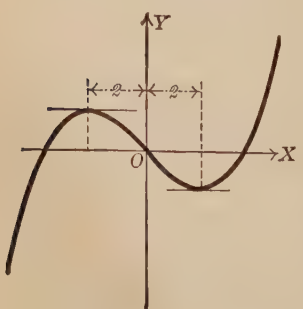

FIG. 9.

EX. 1. Investigate the curve $y = \frac{1}{12}x^3 - x$ by means of its derivative.

Differentiating, we get

$$D_x y = \frac{3x^2}{12} - 1 = \frac{(x+2)(x-2)}{4}.$$

For $x > 2$ or $x < -2$, $D_x y$ is positive, and consequently y increases with x. For values of x between 2 and -2, $D_x y$ is negative, and y therefore decreases as x increases. For $x = 2$ and for $x = -2$, $D_x y = 0$; hence at these points the tangent to the curve is parallel to the X-axis. The curve is shown in Fig. 9.

EXERCISES

Investigate the following curves by means of their derivatives.

1. $y = x^3 - 3x^2 + 6x$. **2.** $y^2 = 8x - 10$. **3.** $y = \frac{a^2}{x}$.

4. For what values of x does the function $x^2 + \frac{1}{x}$ increase with x? For what value does it decrease as x increases?

5. The equation

$$\tau = 53.6\,p^{\frac{1}{5}} - 35.7$$

gives approximately the temperature τ of steam as a function of its pressure p. (Here τ is in degrees Fahrenheit, and p in pounds per square foot.) Show how the derivative $D_p\tau$ changes as the pressure increases, and sketch roughly the curve $\tau = f(p)$.

6. A cylindrical vessel with one end open is to hold 300 cu. in. The superficial area A of this vessel (cylindrical wall plus one base) is a function of the radius of the base. Deduce the equation and examine it by means of the derivative. Interpret the results.

7. The efficiency η of a hoisting device is a function of the load P raised as expressed by the equation

$$\eta = \frac{P}{mP + n},$$

where m and n are constants. Show how the derivative varies with the load P and sketch the general form of the curve $\eta = f(P)$.

8. By means of the derivative investigate the curves:

$$(a)\ y = a + b(x - c)^{\frac{1}{3}};$$

$$(b)\ y = a + b(x - c)^{\frac{2}{3}}.$$

Show the form of the curves in the vicinity of the point (c, a).

9. Given the continuous curve $y = f(x)$. Show by means of the graph that, for positive values of Δx, Δy is positive or negative according as the function is increasing or decreasing at a given point. By considering the limit $\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x}$ deduce the general law given on p. 48.

35. Derived curves. The derivative of a function is also, in general, a function of the independent variable, and may be represented by a graph in the same manner and under the same conditions as the original function. This curve, whose equation is $y = f'(x)$, is known as the **derived curve**.

The principles developed in the last article enable us to establish certain general properties of derived curves. Thus the graph of $y=f'(x)$ crosses the axis of x at those points for which the original curve has a turning point. Moreover, we can tell whether the derived curve crosses the X-axis from above to below or *vice versa*; for we know that as x increases, $f'(x)$ is positive or negative according as $f(x)$ increases or decreases. Hence, if the values of $f(x)$ are increasing as the turning point is approached, and decreasing after it is passed, as at points A and E, Fig. 10, the

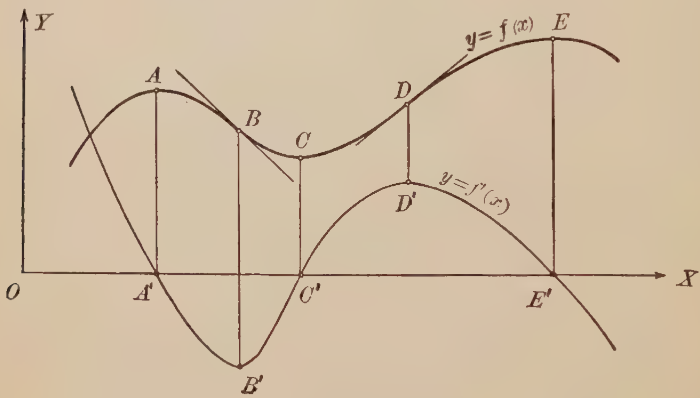

Fig. 10.

values of $f'(x)$ pass from positive to negative, and the derived curve passes from the upper to the lower side of the axis, as shown at points A' and E'. At the turning point C, on the other hand, $f(x)$ changes from a decreasing to an increasing function and $f'(x)$ passes from negative to positive values; that is, the derived curve passes from the lower to the upper side of the X-axis at the point C'.

Where the derived curve has turning points, as at B' and D', the curve $y=f(x)$ has points of inflection, as B and D. See Art. 88.

For those functions that are to be considered in the present volume the derived curve is, in general, a continuous curve. We shall meet, however, two exceptional cases in which it becomes discontinuous.

1. When the tangent to the curve $y=f(x)$ becomes perpendicular to the X-axis, as at M, Fig. 11. Here the value of $f'(x)$ becomes infinite and the derived curve m has infinite branches.

2. When the curve $y=f(x)$ has an angular point, as at N, Fig. 12. At such a point, the limiting position of the tangent as the point of tangency approaches N from the left is different

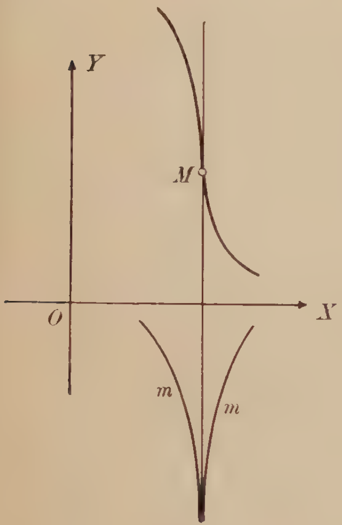

FIG. 11.

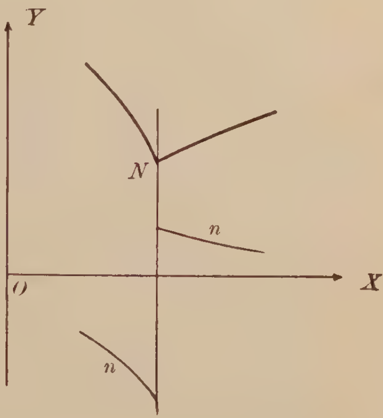

FIG. 12.

from the limiting position as the point of tangency approaches N from the right. Hence, the value of $f'(x)$ takes a sudden jump, and the derived curve n has a discontinuity.

The ordinates of the derived curve are the successive values of D_xy of the original curve. Since D_xy measures the slope of the tangent to the original curve, it follows that for any value of x the ordinate of the derived curve measures the slope of the original curve.

It is evident that the slope of the curve $y=f(x)$ is independent of the position of the axes so long as they remain parallel to their original position. We can shift the curve in the direction of the Y-axis, Fig. 8, and the values of $f'(x)$ will remain unchanged.

In other words, the curves $y=f(x)$ and $y=f(x)+c$, where c is any constant, have the same derived curve (see Art. 24).

EXERCISES

Draw the graphs of the following functions and the derived curve for each of them. Study carefully the combined graphs, and trace out the connection between the slope of the original curve and the ordinate of the derived curve.

1. $y=\frac{x^3}{12}-x.$

2. $y=x^3-2x^2+5.$

3. $y^2=8x-10.$

4. $xy=25.$

5. $y=3+\sqrt[3]{x-4}.$

6. $y=3+(x-4)^{\frac{2}{3}}.$

7. Draw several curves, and by observing the variation in the slope and the turning points sketch in approximately the derived curves.

8. Draw a curve at random, and by observing the variation of the ordinate draw roughly the curve of which the first curve is the derived curve.

36. Rolle's theorem. The following proposition, known as Rolle's theorem, is essential in the development of certain other useful theorems.

THEOREM. *If $f(x)$ and $f'(x)$ are single-valued and continuous for all values of x from $x=a$, to $x=b$, and if $f(a)=f(b)=0$, then $f'(x)$ vanishes for at least one value of x between a and b.**

Either $f(x)$ has a constant value zero for all values of x between a and b, or it varies with x. In the first case $f'(x)$ is zero for all values of x. In the second case, since

$$f(a)=f(b)=0,$$

$f(x)$ must at some point begin to increase and afterwards decrease, or *vice versa*. It must then have a turning point for some particular value of x, say $x=x_1$, lying between a and b, since, by hypothesis, $f(x)$ is continuous.

Geometrically, Rolle's theorem means that if a continuous curve cuts the X-axis in two points $x=a$, $x=b$, and has a definite direction at every point in this interval (a, b), then at some

* Rolle's theorem is stated here in sufficiently general terms for our present purposes. The proof given holds for the theorem as stated. For a more general statement of the theorem and its proof, see Pierpont's *Theory of Functions*, Vol. I, p. 246.

intervening point, say $x = x_1$, the tangent to the curve is parallel to the X-axis (Fig. 13).

It is at once evident that, instead of the condition $f(a) = f(b) = 0$, we might have $f(a)$ and $f(b)$ equal to any constant so long as they are equal to each other. The argument remains precisely the same in the two cases.

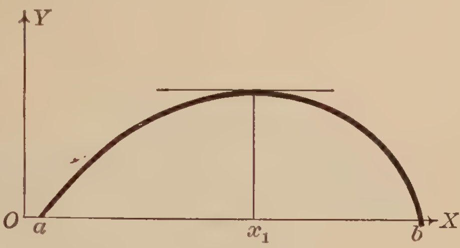

FIG. 13.

EXERCISES

1. By Rolle's theorem show that at least one real root of the equation $f'(x) = 0$ lies between any two real roots of the equation $f(x) = 0$.

2. From Ex. 1 show that if two roots of $f(x) = 0$ are equal, one root of $f'(x) = 0$ coincides with them. Give a geometric illustration of this statement.

37. Law of the mean. By the use of Rolle's theorem, we may deduce one of the most fundamental theorems of the differential calculus, known as the **law of the mean,** or the theorem of mean value.

THEOREM. *Let $f(x)$ and $f'(x)$ be single-valued and continuous functions of x in the interval $a \leqq x \leqq b$. Then there exists at least one value x_0 of x for which*

$$\frac{f(b) - f(a)}{b - a} = f'(x_0), \qquad a < x_0 < b.$$

Without loss of generality, we may assume $f(a) < f(b)$. Geometrically, the quotient $\dfrac{f(b) - f(a)}{b - a}$ gives the slope to the secant AB, Fig. 14. Let $y = f(x)$ be represented by the curve $APDB$, passing through the points A and B; $f'(x_0)$ is then the slope of the tangent to this curve at the point D whose abscissa is x_0. The theorem asserts geometrically that there exists at least one point D of the curve at which the tangent is parallel to the secant line AB.

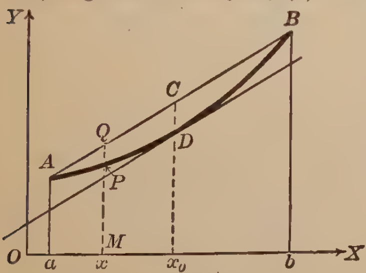

FIG. 14.

To prove this theorem we proceed as follows. The equation of the line AB is of the form

$$y = m(x - a) + c,$$

where the slope m is $\dfrac{f(b) - f(a)}{b - a}$ and the ordinate c of the point where the curve cuts the line $x = a$ is $f(a)$. Hence, we have

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a). \tag{1}$$

The distance of any point P on the curve from the line AB, when measuring along the ordinate through P, is given by

$$\begin{aligned}\psi(x) &= PQ = MQ - MP \\ &= \frac{f(b) - f(a)}{b - a}(x - a) + f(a) - f(x).\end{aligned} \tag{2}$$

Now $\psi(x)$ is a function that satisfies all of the requirements of Rolle's theorem. At some point, say $x = x_0$, of the interval (a, b), we have therefore

$$\psi'(x_0) = \frac{f(b) - f(a)}{b - a} - f'(x_0) = 0, \tag{3}$$

whence

$$\boldsymbol{\frac{f(b) - f(a)}{b - a} = f'(x_0),} \tag{4}$$

as the theorem requires.

The theorem may be stated in another form which is sometimes convenient. The fact that x_0 lies between a and b can be expressed by the relation

$$x_0 = a + \theta(b - a),$$

where θ is some number between 0 and 1. We may also put

$$b - a = h.$$

Equation (4) then takes the form

$$\boldsymbol{\frac{f(a + h) - f(a)}{h} = f'(a + \theta h).} \tag{5}$$

The fraction $\dfrac{f(b) - f(a)}{b - a}$ evidently measures the average rate of increase of the function in the interval $b - a$. Thus, suppose s_1 and s_2 denote distances traversed by a moving point in the

times t_1 and t_2 respectively; the fraction $\frac{s_2 - s_1}{t_2 - t_1}$ gives the average speed of the point for the time interval $t_2 - t_1$, and the law of the mean asserts that at some instant within this interval the actual instantaneous speed of the point is equal to the average or mean speed for the whole interval.

EXERCISES

1. Show geometrically that Rolle's theorem is a special case of the law of the mean.

2. If $f(x) = x^2$, find the value of θ that satisfies relation (5): (*a*) when $a = 6$, $h = 1$; (*b*) when $a = 12$, $h = 4$.

3. If $f(x) = x^3$, find the value of θ in order that (5) shall be satisfied when $a = 4$, $h = 1$.

4. A smooth curve is passed through three points (6, 3), (8, 5), and (10, 8.5), which have been determined experimentally, and the slope of the curve at the intermediate point (8, 5) is desired. Find by the law of the mean an approximate value for this slope.

38. Equation of the tangent and of the normal to a curve. Let (x_1, y_1) be any point on the curve whose equation is

$$y = f(x).$$

As we have seen, the slope of the tangent to the curve at the point (x_1, y_1) is given by substituting the value x_1 for x in the derived function $f'(x)$; that is, for $x = x_1$, $\tan\phi = f'(x_1)$. From analytic geometry, the equation of the tangent at any point (x_1, y_1) on the curve is

$$y - y_1 = m(x - x_1),$$

where m denotes the slope of the tangent at the point in question. Replacing m by $f'(x_1)$, we have for the *equation of the tangent* to the curve in terms of the derivative

$$\boldsymbol{y - y_1 = f'(x_1)\,(x - x_1).} \tag{1}$$

The normal to a curve is perpendicular to the tangent to the curve at the point of tangency. The condition that one straight line shall be perpendicular to another is that the slope of the one shall be the negative reciprocal of that of the other. It follows

that the *equation of the normal* to the given curve at the point (x_1, y_1) is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1). \qquad (2)$$

Because of the relation

$$D_y x = \frac{1}{D_x y} = \frac{1}{f'(x)}, \qquad \text{(Art. 32)}$$

the slope of the normal can frequently be most conveniently obtained by substituting $y = y_1$ in $D_y x$.

Ex. Find the equations of the tangent and normal to the curve $y = x^{\frac{3}{2}}$ at the point (4, 8).

We have $\qquad f'(x) = \frac{3}{2}x^{\frac{1}{2}},$

and for $x = 4$, $\qquad f'(4) = \frac{3}{2} \times 2 = 3.$

Substituting this value of $f'(x)$ in (1), the equation of the tangent is

$$y - 8 = 3(x - 4),$$

or

$$3x - y = 4.$$

At the given point, $\dfrac{1}{f'(x_1)} = \dfrac{1}{3}$. Hence, the equation of the normal is

$$y - 8 = -\tfrac{1}{3}(x - 4),$$

or

$$x + 3y = 28.$$

EXERCISES

Find the equations of the tangent and normal to the following curves at the points indicated.

1. $y^2 = 3x + 10$, at $(-2, 2)$.

2. $xy = 36$, at $(4, 9)$.

3. $y^2 = \dfrac{x^3}{2a - x}$, at (a, a).

4. $(x - 4)^2 + (y + 3)^2 = 100$, at $(10, 5)$.

5. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$, at $(2, \frac{4}{5}\sqrt{21})$.

6. $y = x^3 - 3x + 10$, at $(2, 12)$.

7. $y = x^2 - \dfrac{1}{x}$, at $(2, 3\frac{1}{2})$.

Find the equations of the tangent and normal to the following curves at the point (x_1, y_1).

8. $y^3 = ax^2$.

9. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

10. $x^n + y^n = a^n$.

11. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

39. Lengths of tangent, normal; subtangent, subnormal. The length of the part of the tangent to a curve that lies between the point of tangency and the axis of abscissas is called the **length of the tangent.** The length of that part of the normal intercepted between the same point on the curve and the axis of abscissas is called the **length of the normal.** The projections of these lengths upon the axis of abscissas are called respectively the **subtangent** and the **subnormal.** Thus in Fig. 15 let the tangent to the curve at the point P, having the coördinates x_1, y_1, meet the X-axis in the point A, and let the normal at the point P meet it at C. Let BP be a perpendicular let fall from P upon the X-axis. Then for the point P, PA is the length of the tangent, PC is the length of the normal, and AB and BC are the subtangent and subnormal respectively.

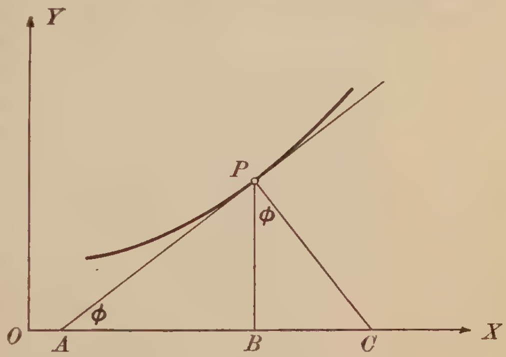

FIG. 15.

In the triangles APB and BPC the side $BP = y_1$ is given, as is also the angle $PAB = CPB = \phi$. If the equation of the curve is $y = f(x)$, $\tan\phi = f'(x_1)$. The lengths of the tangent and normal, the subtangent and the subnormal, may therefore be expressed in terms of y_1 and $f'(x_1)$. For example,

since $$AB = BP \cot\phi,$$

we have $$\text{subtangent} = y_1 \cot\phi = \frac{y_1}{f'(x_1)}.$$

EX. Find the length of the normal to the curve $y^2 = \dfrac{x^3}{4}$ at the point (4, 4).

From Fig. 15, the required length is $y_1 \sec\phi$. Since $D_x y = \frac{3}{4} x^{\frac{1}{2}}$,

we have $$f'(x_1) = f'(4) = \tfrac{3}{4}\sqrt{4} = \tfrac{3}{2};$$

therefore $$\tan\phi = \tfrac{3}{2}, \quad \sec\phi = \sqrt{1 + (\tfrac{3}{2})^2},$$

and the length of the normal is $4\sqrt{1 + (\frac{3}{2})^2} = 2\sqrt{13}$.

EXERCISES

1. Find the length of the tangent, the subtangent, and the subnormal for the curve $y^2 = \frac{x^3}{4}$ at the point (4,4).

2. Derive general formulas for the length of the tangent, length of the normal, subtangent, and subnormal in terms of y_1 and $f'(x_1)$.

Find the subtangent, subnormal, length of the tangent, and length of the normal for each of the following curves at the points indicated.

3. $y^2 = 8x$, at (2, 4).

4. $xy = a^2$, at (x_1, y_1).

5. $(x-4)^2 + (y+3)^2 = 25$, at (7, 1).

6. $y = x^3 - 3x + 10$, at (x_1, y_1).

7. $y^3 = ax^2$, at (a, a).

8. $y^2 = \frac{x^3}{2a - x}$, at (a, a).

9. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, at (x_1, y_1).

10. Show that the subnormal to the parabola $y^2 = 2mx$ is constant.

11. Show that the length of the normal to the curve $x^2 + y^2 = a^2$ is constant.

12. Using the equations of the tangent and the normal in Art. 38, derive formulas for the length of the tangent, length of the normal, etc., by finding the intercepts of these lines on the X-axis.

40. Tan ψ, cot ψ. Let ρ, θ be the polar coördinates of any point on a curve, θ being taken as the independent variable. In order to determine the direction of the curve at any point, it is convenient to express the tangent and cotangent of the angle ψ between the radius vector and the tangent to the curve in terms of ρ, θ, and their derivatives.

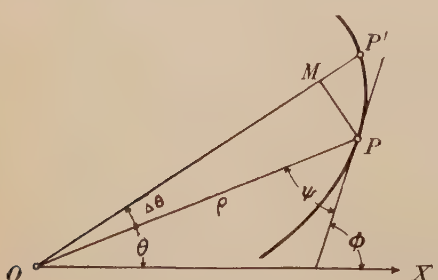

FIG. 16.

In Fig. 16, let P be any point on the given curve, and let its polar coördinates be ρ, θ. If θ takes an increment $\Delta\theta$, then ρ will have a corresponding increment $\Delta\rho$. Let P' be the

point having the coördinates $\theta + \Delta\theta$, $\rho + \Delta\rho$, and let MP be drawn perpendicular to OP'.

From the figure, we have

$$MP = \rho \sin \Delta\theta, \quad OM = \rho \cos \Delta\theta,$$

and $$MP' = OP' - \rho \cos \Delta\theta = \rho + \Delta\rho - \rho \cos \Delta\theta.$$

Hence, $$\tan MP'P = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

As $\Delta\theta \doteq 0$, the angle $MP'P$ approaches the angle ψ. Hence we have

$$\underset{\Delta\theta \doteq 0}{L} \tan MP'P = \underset{\Delta\theta \doteq 0}{L} \frac{\rho \sin \Delta\theta}{\Delta\rho + \rho(1 - \cos \Delta\theta)},$$

or $$\tan \psi = \underset{\Delta\theta \doteq 0}{L} \frac{\rho \dfrac{\sin \Delta\theta}{\Delta\theta}}{\dfrac{\Delta\rho}{\Delta\theta} + \rho\left(\dfrac{1 - \cos \Delta\theta}{\Delta\theta}\right)}.$$

But we have seen that

$$\underset{\Delta\theta \doteq 0}{L} \frac{\sin \Delta\theta}{\Delta\theta} = 1, \text{ and } \underset{\Delta\theta \doteq 0}{L} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0; \quad \text{(Art. 13)}$$

hence, we have

$$\boldsymbol{\tan \psi = \frac{\rho}{D_\theta \rho} = \rho D_\rho \theta.} \tag{1}$$

From (1) it follows at once that

$$\boldsymbol{\cot \psi = \frac{1}{\tan \psi} = \frac{1}{\rho} D_\theta \rho.} \tag{2}$$

Ex. Let the equation of a curve be $\rho^2 = \dfrac{a}{\theta}$. Find the angle ψ for $\rho = 1$.

Writing the given equation in the form

$$\theta = \frac{a}{\rho^2} = a\rho^{-2},$$

we find $$D_\rho \theta = -2\,a\rho^{-3}.$$

Hence, $$\tan \psi = \rho D_\rho \theta = -2\,a\rho^{-2} = -2\frac{a}{\rho^2},$$

and $$\cot \psi = -\frac{\rho^2}{2\,a}.$$

For $\rho = 1$, $$\tan \psi = -2\,a,$$

whence $$\psi = \text{arc tan}\,(-2\,a).$$

41. Length of polar tangent and polar normal; polar subtangent, polar subnormal. Given a curve and the polar coördinates (ρ_1, θ_1) of any point P upon it (Fig. 17). At the point P, draw the tangent and the normal to the curve and through the pole O draw a perpendicular to the radius vector OP. Let this perpendicular cut the tangent and the normal in the points T and N respectively. PN is called the **length of the polar normal,** and PT the **length of the polar tangent.** The projections of these lines upon the perpendicular, namely, NO and OT, are called the **polar subnormal,** and **polar subtangent,** respectively.

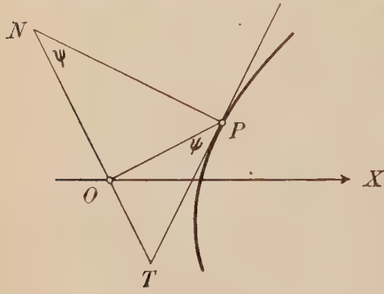

FIG. 17.

It will be observed that in the case of rectangular coördinates, the line upon which the tangent and normal were projected to determine the subtangent and subnormal was the X-axis, while in polar coördinates, it is not the initial line but a line through the pole perpendicular to the radius vector.

In each of the right triangles OPT and OPN one angle ψ and one side $OP = \rho_1$ are given. Since, from the preceding article, $\psi = \text{arc tan } \dfrac{\rho_1}{F'(\theta_1)}$, where $F'(\theta_1)$ is obtained from the equation of the curve $\rho = F(\theta)$, it follows that the four lengths in question can be expressed in terms of ρ_1 and $F'(\theta_1)$.

Ex. Find the lengths of the polar tangent of the curve $\rho^2 = \dfrac{a^2}{\theta}$.

Putting the equation in the form $\rho = a\theta^{-\frac{1}{2}}$,

we get $$D_\theta\rho = -\tfrac{1}{2}a\theta^{-\frac{3}{2}}; \text{ hence, } F'(\theta_1) = -\frac{a}{2\,\theta_1^{\frac{3}{2}}}.$$

Therefore $$\cot\psi = -\frac{1}{\rho_1}\,\frac{a}{2\,\theta_1^{\frac{3}{2}}} = -\frac{\theta_1^{\frac{1}{2}}}{a}\,\frac{a}{2\,\theta_1^{\frac{3}{2}}} = -\frac{1}{2\,\theta_1},$$

and $$\tan\psi = -2\,\theta_1.$$

From Fig. 17, length of polar tangent $= \rho_1 \sec\psi = \dfrac{a}{\theta_1^{\frac{1}{2}}}\sqrt{1 + 4\,\theta_1^2}.$

EXERCISES

1. Derive general formulas for the lengths of the polar tangent and polar normal, the polar subtangent, the polar subnormal, in terms of ρ_1 and $F'(\theta_1)$.

2. Find the polar subtangent and polar subnormal of the curve $\rho^2 = \frac{a^2}{\theta}$.

3. Find the length of the polar normal of the same curve.

Find general expressions for the length of the tangent, subtangent, length of the normal, and subnormal of the following curves:

4. $\rho = a\theta$. 5. $\rho = a\theta^2$. 6. $\rho = 1 + \frac{1}{\theta}$. 7. $\rho^2 = a\theta + b\theta^2$.

8. Find $\tan\psi$ for the curves of Exs. 4–7.

9. Show that the polar subtangent to the curve $\rho\theta = a$ is of constant length. Trace the curve.

42. Speed and acceleration. Let a point move in a given path, straight or curved, and let s denote the variable distance of the point measured along the path from some fixed origin O. Assuming continuous motion from O, the distance s evidently depends upon the time t; that is, $s = f(t)$. In a time interval Δt the point moves over an element of path of length Δs, and the ratio $\frac{\Delta s}{\Delta t}$ gives the **mean speed** of the point for this interval. The limiting value of this quotient as the time interval Δt approaches zero gives the **instantaneous speed** at the beginning of the interval. Denoting this by v, we have therefore

$$v = \underset{\Delta t \doteq 0}{L} \frac{\Delta s}{\Delta t} = D_t s; \tag{1}$$

that is, *the speed of a moving point is the time-derivative of the space traversed.*

The words "speed" and "velocity" are often used as synonyms. An important distinction is, however, frequently made. The term "speed" is used to indicate merely the rate of motion in the path irrespective of direction. Speed, therefore, has only magnitude. The term "velocity" carries with it the additional notion of direction, hence to specify a velocity both magnitude and direction are required.

The speed v is, in general, a function of the time t. If Δv denotes the change of speed between two points, then the quotient

$\frac{\Delta v}{\Delta t}$ defines the **average tangential acceleration*** between those points; and the limit of this quotient as $\Delta t \doteq 0$ defines the **instantaneous tangential acceleration.** Denoting this by a, we have

$$a = \underset{\Delta t \doteq 0}{L} \frac{\Delta v}{\Delta t} = D_t v; \tag{2}$$

that is, *the tangential acceleration is the time-derivative of the speed.*

The ordinary unit of speed is the foot per second, that of acceleration the foot per second per second. These are abbreviated to ft./sec. and ft./sec.^{2} respectively.

Ex. In a certain motion the space described is expressed as a function of the time by the following equation:

$$s = at^2 + bt + c.$$

For the speed, we have $\quad v = D_t s = 2\,at + b,$

and for the tangential acceleration, we obtain

$$a = D_t v = 2\,a.$$

43. Angular speed and acceleration. If a body rotates about a fixed axis, any given point of it, not on the axis, moves in a circle whose center lies on this axis, and whose plane is perpendicular to the axis. Let O be the center and PAB the circular path of the point, Fig. 18. During the motion of the point from P to A, the radius OA sweeps over an angle θ, and in the additional interval of time Δt required for the motion from A to B it sweeps over the angle $AOB = \Delta\theta$. The ratio $\frac{\Delta\theta}{\Delta t}$ defines the **mean angular speed** of the body between the positions OA and OB, and the limit of this

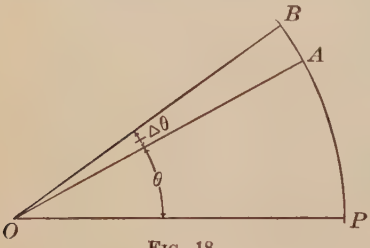

FIG. 18.

---

* Tangential acceleration is the acceleration in the direction of the motion, that is, tangent to the path. In the case of rectilinear motion, this is the only acceleration; but in the case of a point moving in a curve, there is another acceleration perpendicular to the tangent.

ratio as B is made to approach A defines the **instantaneous angular speed** for the position OA of the radius. Denoting this angular speed by ω, we have

$$\omega = \underset{\Delta t \doteq 0}{L} \frac{\Delta \theta}{\Delta t} = D_t \theta; \tag{1}$$

that is, *the angular speed is the time-derivative of the angle swept over.*

The angular speed may be constant or variable. If variable, the increment between two positions, say A and B, may be denoted by $\Delta\omega$, and the ratio $\dfrac{\Delta\omega}{\Delta t}$ gives the **mean angular acceleration** between the positions in question. The limit of this mean value as the two chosen positions are made to approach each other is the **instantaneous angular acceleration**, which is denoted by α. We have then

$$\alpha = \underset{\Delta t \doteq 0}{L} \frac{\Delta \omega}{\Delta t} = D_t \omega; \tag{2}$$

that is, *the angular acceleration is the time-derivative of the angular speed.*

With angles measured in radians, the unit of angular speed is the radian per second (rad./sec.), and that of angular acceleration is the radian per second per second (rad./sec.^{2}).

The relation between the angular speed of a rotating body and the linear speed of any point of the body in its circular path is readily derived. Referring to Fig. 18, AB is an element Δs of the circular path of P. Denoting the radius OA by r, we have therefore

$$\Delta s = r\Delta\theta,$$

whence

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t},$$

and finally

$$D_t s = rD_t\theta. \tag{3}$$

That is, *the speed of any point of a body rotating about a fixed axis is the product of the distance of the point from the axis and the angular speed of the body.*

A similar law holds for tangential acceleration. Thus from (3)

$$v = r\omega,$$

whence $$D_t v = r D_t \omega,$$

or $$a = r\alpha; \tag{4}$$

that is, *the tangential acceleration of the point is the angular acceleration about the fixed axis multiplied by the distance from the point to the axis.*

EXERCISES

1. If a body is projected vertically upward with a speed of v_0 feet per second, the space traversed in t seconds from the instant of projection is given by the equation

$$s = v_0 t - \tfrac{1}{2} g t^2.$$

(a) Find expressions for the speed and acceleration at the time t_1. (b) Take $v_0 = 300$ and $g = 32.2$, and find the speed and acceleration at the end of 2 seconds; (c) at the end of 12 seconds.

2. In Ex. 1, find the whole time occupied by the body rising and falling, also the height to which it rises.

SUGGESTION: Make $s = 0$ and solve for t.

3. When a body moves in a straight line under the influence of an attractive force that varies as the inverse square of the distance, the motion is given by the equation $v^2 = \frac{k}{s}$, in which k is a constant. Show that the acceleration is $-\frac{k}{2s^2}$.

4. The angle (in radians) through which a given rotating body turns, starting from rest, is given by the equation

$$\theta = t^2 + 5t.$$

Find the angular speed and angular acceleration at the end of 4 seconds.

5. If the angle is given by the equation

$$\theta = 112t - 16t^2,$$

find (a) the speed and acceleration at the end of 2.5 seconds; (b) the time that elapses before the body comes to rest.

Derive expressions for ω and α from each of the following relations between θ and t:

6. $\theta = at - bt^3$. **7.** $\theta = at^{\frac{1}{2}}$. **8.** $\theta = a + bt + ct^2$.

44. Miscellaneous applications. In physics and in chemistry the notion of the derivative is repeatedly encountered. In

the case of a moving point, the term "speed" is used for the time-derivative of the distance traversed. In chemistry, likewise, we have the same term used for other time-derivatives; thus, the speed of reaction and the speed of solution are such derivatives. In physics we meet with a great number of derivatives expressing the rates of change of various physical magnitudes. The heating of substances, variations of pressure, density, and temperature, the variations in velocity and in energy, etc., are such changes. A few of these derivatives are discussed in the following paragraphs.

(*a*) *Coefficients of expansion.* Let a rod or wire have unit length (1 foot or 1 yard) at some standard temperature, say 32° F. or 0° C. When heated the rod expands by an amount x depending upon the temperature. Thus, denoting the temperature by τ, the expansion is $x = f(\tau)$ and the new length of the rod is

$$1 + x = 1 + f(\tau). \tag{1}$$

If now the temperature rises by an amount $\Delta\tau$, the length of the rod will increase by a corresponding amount Δx. The quotient $\dfrac{\Delta x}{\Delta \tau}$ is called the *average coefficient of linear expansion* for the interval $\Delta\tau$, and its limit as $\Delta\tau$ is made to approach zero is the *coefficient of linear expansion for the temperature τ*. Denoting this by C_1, we have

$$C_1 = \underset{\Delta\tau \doteq 0}{L} \frac{\Delta x}{\Delta \tau} = D_\tau x. \tag{2}$$

In most cases we may assume with sufficient approximation

$$x = f(\tau) = a\tau + b\tau^2,$$

whence

$$C_1 = a + 2\,b\tau; \tag{3}$$

that is, C_1 is itself a function of the temperature.

In the same way we may arrive at the coefficients of superficial and cubical expansion. Consider a cube with its edge having unit length at 0° C. Assuming equal expansion in all directions, each edge at the temperature τ will have a length $1 + f(\tau)$. Hence the area of a face will be

$$A = [1 + f(\tau)]^2, \tag{4}$$

and the volume of the cube will be

$$V = [1 + f(\tau)]^3. \tag{5}$$

The derivatives $D_\tau A$ and $D_\tau V$ are the *coefficients of superficial and cubical expansion* respectively, and may be denoted by C_2 and C_3.

(*b*) *Specific heat.* A body, originally at some definite temperature τ_0, is heated and its temperature rises. The heat Q absorbed by the body is in general a function of the final temperature τ; that is,

$$Q = f(\tau). \tag{6}$$

An increment of heat ΔQ causes a corresponding rise in temperature $\Delta\tau$, and the quotient $\frac{\Delta Q}{\Delta\tau}$ is called the *average specific heat* of the body for the interval $\Delta\tau$. The limit of this quotient as $\Delta\tau$ approaches zero is defined as the *specific heat at the temperature* τ. Denoting this by c, we have

$$c = \underset{\Delta\tau \doteq 0}{L} \frac{\Delta Q}{\Delta\tau} = D_\tau Q. \qquad (7)$$

EXERCISES

1. Regnault's experiments on the heating of various substances are represented by the following equations:

Ether,	$Q = 0.5290\,\tau + 0.000296\,\tau^2.$
Chloroform,	$Q = 0.2324\,\tau + 0.00005\,\tau^2.$
Bisulphide of carbon,	$Q = 0.2352\,\tau + 0.000082\,\tau^2.$

In each case Q denotes the heat required to raise the temperature of the substance from $0°$ C. to $\tau°$ C. For each substance find the specific heat at $20°$ C.

2. It is found by experiment that the volume of water which at $4°$ C. has unit volume is given by the equation

$$V = 1 + a(\tau - 4)^2,$$

where τ denotes the temperature of the water and $a = 0.00000838$. Find the coefficient of cubical expansion when $\tau = 0°$; also for $\tau = 20°$.

3. The electrical resistance R of a wire varies with the temperature τ of the wire, the relation being expressed by $R = f(\tau)$. (*a*) What is expressed by the derivative $D_\tau R$? (*b*) Find the expression for this derivative from Callendar's formula

$$R = R_0(1 + \alpha\tau + \beta\tau^2)$$

in which R_0, α, and β are constants.

4. (*a*) What derivative expresses the rate of the rise of temperature of a gas with respect to the pressure? (*b*) With respect to the volume? (*c*) What derivative gives the rate of change of the energy U of the gas with respect to the temperature τ?

5. The pressure of the atmosphere decreases as the distance from the earth's surface increases, and the rate of change of pressure with the height is proportional to the pressure. State this law in the form of an equation.

MISCELLANEOUS EXERCISES

1. The equation of a curve is $4a^2y = x^3 - 2ax^2 + a^3$.

(*a*) Find the slope for $x = 0$ and $x = a$.

(*b*) Find the points where the curve is parallel to the X-axis.

(*c*) Find the points at which the slope of the curve is 1.

2. Find the angle at which the circle $x^2 + y^2 = 8x$ intersects the curve $y^2 = \frac{x^3}{2-x}$.

3. Find the equation of the tangent to the hyperbola $xy = 30$ that cuts the X-axis in the point $x = 4$. Show that the point of tangency of *any* tangent to this curve lies midway between the intersections of the tangent with the X- and Y-axes.

Investigate the following curves by means of their derivatives:

4. $y = x^2(a-x)^2$. **5.** $y = \frac{x}{1+x^2}$. **6.** $y = \frac{x(x-2)}{2x-5}$.

7. Draw curves showing the speed v of a body falling in vacuo ($s = \frac{1}{2}gt^2$), (a) with values of t as abscissas, (b) with values of s as abscissas. Show that the subnormal of the second curve is numerically equal to the acceleration.

8. In the case of the semicubical parabola $ay^2 = x^3$ show that

$$\text{subnormal} = (\text{subtangent})^2 \cdot \frac{27}{8a}.$$

9. (a) Find the equation of a curve whose polar subnormal is constant. (b) Find a curve whose polar subtangent is constant.

10. Show that Rolle's theorem does not hold for $f(x) = (x-1)^{\frac{2}{3}} - 1$ between $x = 0$ and $x = 2$. Explain why.

11. Apply Eq. (5), Art. 37, to the function $f(x) = x^3 - 6x - 8$. Find the value of θ for $a = 4$, $h = 1$; also for $a = 7$, $h = 2$.

12. Show that for every quadratic function $f(x) = ax^2 + bx + c$, Eq. (5), Art. 37, is satisfied when $\theta = \frac{1}{2}$. From this fact deduce a geometric property of the parabola.

13. If Q denotes quantity of heat, τ temperature, s length or distance, and t time, express in words what is meant by the equation

$$D_tQ = -k \cdot D_s\tau,$$

which applies to the flow of heat along a bar.

14. Show that the coefficients of superficial and cubical expansion are respectively two and three times the coefficient of linear expansion, very nearly.

15. Let m denote the mass of a body moving in a straight line, v the speed, and a the acceleration of the body, F the constant force acting on it, and s the distance traversed. From mechanics we have the following definitions and symbols:

$$mv = \text{momentum of body, } (M),$$
$$ma = \text{force acting, } (F),$$
$$Fs = \text{work of force } F, \ (W).$$

Show that $F = D_t(mv) = D_tM$,

and also that $F = D_sW$.

16. The kinetic energy of the body (Ex. 15) is given by the expression

$$T = \tfrac{1}{2} mv^2.$$

(*a*) Show that the momentum is the v-derivative of the kinetic energy. (*b*) Show that the time-derivative of the kinetic energy is the product Fv.

17. If for *any* motion a curve is drawn with the speed v as ordinates and the distances s as abscissas, show that the tangential acceleration for any value of s is given by the subnormal to the curve at that point.

18. The equation of van der Waals,

$$\left(p + \frac{a}{v^2}\right)(v - b) = C,$$

gives the isothermal curves of carbon dioxide. By means of the derivative investigate the general form of the isothermal curves obtained by giving C different constant values.

19. The following values of corresponding pressures and temperatures of saturated steam are taken from a standard table:

p lb./sq. in.	84	85	86	87	88
τ temperature F.	315.19°	316.02°	316.84°	317.65°	318.45°

Making use of the law of the mean, find approximately the value of the derivative $D_p\tau$ for $p = 85$; also for $p = 87$. Take $\theta = 0.5$.

20. Rankine's formula for long columns has the form

$$y = \frac{a}{1 + bx^2}.$$

By means of the derivative investigate the general character of this function, and sketch the curve that represents it.

CHAPTER IV

THE DIFFERENTIAL NOTATION

45. The derivative as a rate. The fundamental problem of differential calculus is the measurement of the rate of change of the function with respect to the variable. We are not concerned so much with the actual change of the function as with its change per unit increase of the variable, or, in other words, its *rate* of change. The great importance of the derivative lies in the fact that it gives a precise measure of this rate of change. Thus the derivative $D_t s$ measures the rate of change of the distance s with respect to the time, or, more briefly stated, the time-rate of s; the derivative $D_t\theta$, the angular speed, is likewise the time-rate of the angle θ. Again, consider the case of heating a metal bar. The length x of the bar is a function of its temperature τ, say $x = f(\tau)$, and the derivative $D_\tau x$ gives the rate of change of length with respect to the temperature.

That the derivative measures the rate of change of the function is easily seen. If Δy denotes the change of the function corresponding to a change Δx of the variable, the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

gives the average rate of change for the interval Δx. The rate at which y is changing with respect to x at any particular value of x, as x_0, is the limiting value of this quotient as Δx approaches zero, that is,

$$\underset{\Delta x \doteq 0}{L} \frac{\Delta y}{\Delta x} = f'(x_0).$$

It is not necessary that y be expressed directly in terms of x in order that we may discuss their comparative rates of change. In fact, it is often convenient to compare the rates of change of two functions by means of a third variable. For example, we may

compare the relative rate of change in the market value of wheat in Chicago and steel rails in New York by the use of a third variable representing money. The speed of a locomotive may be compared with that of a street car, or the relative speed of two chemical reactions may be found by means of a third variable representing time.

The conception of the derivative as a rate is fundamental in the developments of the present chapter.

46. Differentials. In the preceding chapters we have employed the notation of derivatives explained in Art. 16. We shall now introduce another notation, that of differentials, which for certain purposes is more convenient than that of derivatives.

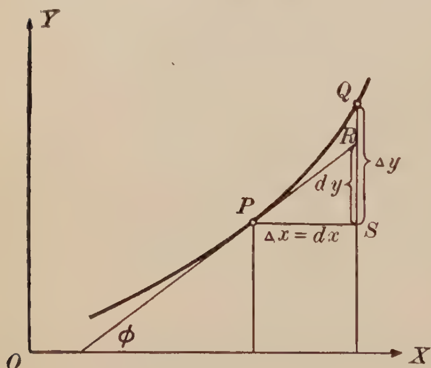

FIG. 19.

Let us consider first the case in which y is expressed directly in terms of x, and let $y = f(x)$ be a continuous function represented by the curve in Fig. 19. At any point P of the curve let a tangent be drawn, and let PS represent an assumed increment Δx of the independent variable x. The corresponding increment Δy of the function is represented by SQ, while the intercept SR between PS and the tangent represents the product $\Delta x \cdot \tan \phi = f'(x)\, \Delta x$. Denoting this product by dy, we have therefore

$$dy = f'(x)\, \Delta x. \tag{1}$$

In order to make the notation symmetrical, it is customary to denote the increment of x by dx; that is, to write dx for Δx. With this convention (1) becomes

$$dy = f'(x)\, dx. \tag{2}$$

In equation (2) dx is called the **differential** of x, dy the differential of y, and $f'(x)\, dx$ the differential of $f(x)$. We have therefore the

following definitions: *The differential of the variable x is merely an assumed increment of x; and the differential of the function y is the product of the derivative $f'(x)$ of the function and the differential dx of the variable.*

Frequently $f'(x)$ is called the **differential coefficient**, that is, the coefficient of the differential of the independent variable.

Having assigned a value to dx ($\equiv \Delta x$), the values of dy and Δy are determined by the nature of the functional relation $y=f(x)$. Usually these two symbols do not represent the same value. The value of dy is determined by $f'(x)\,dx$, that is, by $dx \cdot \tan\phi$, and is represented in the figure by SR. On the other hand, Δy is the change in the function $f(x)$ corresponding to the change Δx of the independent variable, that is, SQ in the figure. The two symbols dy and Δy represent the same value when, and only when, $y=f(x)$ is represented by a straight line.

It should be noted that dy is the increment that the ordinate *would have* if the rate of change of y at the point P were maintained throughout the interval Δx. It is evident also that if Δx is sufficiently small dy is an approximation to the increment Δy, and that the smaller Δx is chosen, the closer is the approximation.

The relation (2) between dy and dx evidently holds when x and y are given as functions of a third variable t. The parametric representation has the advantage that it enables us to show clearly the relation between differentials and rates as measured in terms of a common variable. Suppose we have given

$$x=\phi(t),\ y=\psi(t). \tag{3}$$

Eliminating t in the second equation by means of the first, we have

$$y=f(x), \text{ where } x=\phi(t). \tag{4}$$

Hence by Art. 31, we have

$$D_t y=f'(x)\,D_t x. \tag{5}$$

As we have seen, the derivatives $D_t x$, $D_t y$ express the rates of change of the variables x and y with respect to t, the common variable in terms of which both x and y are expressed. It will be observed that $D_t x$, $D_t y$ enter homogeneously into equation (5)

as do dx, dy in equation (2). Moreover, by a comparison of equations (2) and (5), we have

$$f'(x) = \frac{dy}{dx} = \frac{D_t y}{D_t x}. \tag{6}$$

We may then say that the rates $D_t x$, $D_t y$ are either equal to the differentials dx, dy or proportional to them. This relation may be, and often is, made the basis of the definition of differentials; and it enables us to write the derivatives $D_t x$, $D_t y$ in terms of the differentials, or *vice versa*, whenever it suits our purpose to do so.

Since we have

$$D_x y \equiv f'(x) = \frac{dy}{dx}, \tag{7}$$

it follows that *a derivative is equal to the quotient of two differentials.* It is frequently convenient to write the derivative as such a quotient, and in the future we shall employ either $D_x y$ or $\frac{dy}{dx}$ as best suits the problem under discussion. We may read $\frac{dy}{dx}$ either "the derivative of y with respect to x," or "differential y divided by differential x."

The distinction between the symbols $D_x y$ and $\frac{dy}{dx}$ should, however, be carefully noted. While both lead to the same numerical result and may therefore be used interchangeably, $D_x y$ indicates that a certain operation has been performed upon the function y with respect to the independent variable x. It is not possible to separate this symbol into two parts. On the other hand, $\frac{dy}{dx}$ is merely a quotient which can be dealt with by the ordinary rules of algebra.

47. Kinematic interpretation of differentials. An instructive illustration of the use of differentials is afforded by the velocity components of a moving point.

Suppose a point P to move along a curve m, Fig. 20, and let P_x and P_y be the projections of P on the X-axis and Y-axis respectively. As P moves on the curve, the projections P_x and P_y move on the axes. The velocity of P_x along OX is evidently the time-

rate of change of the abscissa x, and is therefore given by the derivative $D_t x$. Likewise, the velocity of P_y along the Y-axis is $D_t y$. The velocity of P along the curve has the direction of the tangent PT, and its magnitude is given by the derivative $D_t s$. Suppose that the velocity of P is such that in a unit of time P would be carried along the tangent from P to T. This displacement PT can be effected in the following manner: Move P along a horizontal line from P to A, and at the same time move the horizontal line PA vertically to BT. Evidently, if the horizontal and vertical motions are made with constant velocity, the point P will move along PT with constant velocity. Since the motion takes place in a unit of time, the displacements PT, PA, and PB respresent respectively the velocity in the path, the velocity of the horizontal motion of P_x, and the velocity of the vertical motion of P_y; that is, $PT = D_t s$, $PA = D_t x$, and $PB = D_t y$.

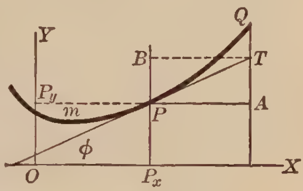

FIG. 20.

The two velocities $D_t x$ and $D_t y$, which together may replace the velocity $D_t s$ in the curve, are called the **component velocities** in the direction of the axes. From the figure the following relations between the velocity in the curve and the components along the axes are evident:

$$\overline{PT}^2 = \overline{PA}^2 + \overline{PB}^2,$$

hence $$(D_t s)^2 = (D_t x)^2 + (D_t y)^2. \qquad (1)$$

Substituting the differentials ds, dx, and dy for the time-rates, we have

$$\boldsymbol{ds^2 = dx^2 + dy^2}. \qquad (2)$$

Furthermore, since $$PA = PT\cos\phi,$$

and $$PB = PT\sin\phi,$$

we have $$\left.\begin{aligned} dx &= ds\cos\phi \\ dy &= ds\sin\phi \end{aligned}\right\}. \qquad (3)$$

EX. 1. A point moves in the parabola $y^2 = 8x$ with a constant velocity of 5 units. Find the components along the axes when the point is at (12.5, 10).

From the given equation, we get

$$2\,yD_t y = 8\,D_t x,$$

or
$$2\,y\,dy = 8\,dx,$$

whence
$$dy = \frac{4}{y}\,dx.$$

But
$$ds = \sqrt{dx^2 + dy^2} = 5,$$

whence
$$dx\sqrt{\frac{16}{y^2} + 1} = 5.$$

We have then
$$dx = \frac{5\,y}{\sqrt{16 + y^2}},$$

which becomes for $y = 10$,
$$dx = \frac{50}{\sqrt{116}} = \frac{25}{29}\sqrt{29};$$

and
$$dy = \frac{4}{10}\cdot\frac{25}{29}\sqrt{29} = \frac{10}{29}\sqrt{29}.$$

If a point moves along a curve whose equation is given in polar coördinates, it is convenient to resolve the velocity along the curve into components along and perpendicular to the radius vector, respectively. Thus, in Fig. 21, the velocity represented by PT may be resolved into components represented by PA and PB. As before, $PT = D_t s$. The velocity component PA is evidently the time-rate of increase of the radius vector $OP = \rho$, that is, $PA = D_t\rho$. The component PB is the velocity that P would have if it remained at rest on the radius while the latter rotated about the pole O; hence, if ω denotes the angular speed of OP about the pole O,

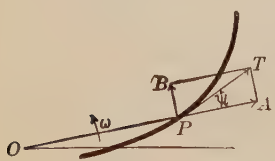

FIG. 21.

$$PB = \overline{OP}\,\omega = \rho\,D_t\theta. \qquad \text{(See Art. 43)}$$

Since
$$\overline{PT}^2 = \overline{PA}^2 + \overline{PB}^2,$$

$$(D_t s)^2 = (D_t\rho)^2 + (\rho\,D_t\theta)^2, \qquad (4)$$

or, substituting differentials for the time-derivatives,

$$ds^2 = d\rho^2 + \rho^2\,d\theta^2. \qquad (5)$$

It follows also from the figure that

$$\left.\begin{aligned}\tan\psi &= \frac{\rho\,d\theta}{d\rho},\\ d\rho &= ds\cos\psi,\\ \rho\,d\theta &= ds\sin\psi.\end{aligned}\right\} \tag{6}$$

Ex. 2. A point moves in the curve $\rho = a\theta$ with a constant velocity m. Find the velocity components when $\theta = \pi$.

Since $$\rho = a\theta,$$

we have $$D_t\rho = aD_t\theta,$$

or $$d\rho = a\,d\theta.$$

Then $$d\rho^2 = a^2\,d\theta^2,$$

$$ds^2 = d\rho^2 + \rho^2\,d\theta^2 = (a^2+\rho^2)\,d\theta^2,$$

$$ds = m = (a^2+\rho^2)^{\frac{1}{2}}\,d\theta,$$

or $$d\theta = \frac{m}{\sqrt{a^2+\rho^2}},$$

whence $$\rho\,d\theta = \frac{m\rho}{\sqrt{a^2+\rho^2}},$$

and $$d\rho = \frac{ma}{\sqrt{a^2+\rho^2}}.$$

For $\theta = \pi$, $\rho = a\pi$. Substituting, we obtain

$$\rho d\theta = \frac{m\pi}{\sqrt{1+\pi^2}}, \text{ velocity perpendicular to } OP,$$

$$d\rho = \frac{m}{\sqrt{1+\pi^2}}, \text{ velocity along } OP.$$

48. Differentiation with differentials. The passage from the derivative to the differential notation, or *vice versa*, is effected by the defining equation

$$dy = f'(x)\,dx. \tag{1}$$

Thus, if $$y = ax^2 + b,$$

$$f'(x) = 2\,ax,$$

whence $$dy = 2\,ax\,dx.$$

Conversely, if we have given

$$dy = (3x^2 + 4)\,dx,$$

we obtain by division,

$$\frac{dy}{dx} = f'(x) = 3x^2 + 4.$$

In the operation of differentiation, we may therefore employ either of the following methods: (1) we may obtain the derivative directly by the theorems already established; or, (2) we may derive an equation in the form (1) which is homogeneous in the differentials of the variables, and then obtain the derivative by division.

The general theorems of differentiation may be easily stated in terms of the differential notation. All that is necessary is to replace D_xy, D_xu, D_xv, etc., by their equivalents $\frac{dy}{dx}$, $\frac{du}{dx}$, $\frac{dv}{dx}$, etc., and multiply by dx. The following are the results thus obtained, which the student may easily verify.

(*a*) Given $y = c$, $\qquad dy = 0.$

(*b*) Given $y = x$, $\qquad dy = dx.$

(*c*) Given $y = (u + v + w)$, $\qquad dy = du + dv + dw.$

(*d*) Given $y = u + c$, $\qquad dy = du.$

(*e*) Given $y = uv$, $\qquad dy = u\,dv + v\,du.$

(*f*) Given $y = cu$, $\qquad dy = c\,du.$

(*g*) Given $y = \frac{u}{v}$, $\qquad dy = \frac{v\,du - u\,dv}{v^2}.$

(*h*) Given $y = \frac{c}{v}$, $\qquad dy = \frac{-c\,dv}{v^2}.$

(*i*) Given $y = u^n$, $\qquad dy = nu^{n-1}\,du.$

The following examples will illustrate the use of the preceding formulas:

Ex. 1. $y = ax\sqrt{1 - x^2}.$

$$\begin{aligned} dy &= \sqrt{1 - x^2}\,d(ax) + ax \cdot d\sqrt{1 - x^2} \\ &= a\sqrt{1 - x^2}\,dx - \frac{ax^2\,dx}{\sqrt{1 - x^2}} \\ &= \frac{a(1 - 2x^2)\,dx}{\sqrt{1 - x^2}}. \end{aligned}$$

Ex. 2. $y = \dfrac{x}{\sqrt{x^2 - a^2}}$.

$$dy = \frac{\sqrt{x^2 - a^2}\,dx - x \cdot d\sqrt{x^2 - a^2}}{(\sqrt{x^2 - a^2})^2}$$

$$= \frac{\sqrt{x^2 - a^2}\,dx - \dfrac{x^2\,dx}{\sqrt{x^2 - a^2}}}{x^2 - a^2} = -\frac{a^2\,dx}{\sqrt{(x^2 - a^2)^3}}.$$

EXERCISES

Differentiate the following functions and express the results in differential form.

1. $y = \dfrac{x - 3}{x - 5}$.

2. $y = \dfrac{x\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}}$.

3. $y = x^{\frac{1}{3}}(x - a)^{-\frac{1}{3}}$.

4. $p = \dfrac{c}{v^m}$.

5. $y = (a + bx^n)^m$.

6. $y = (1 - x)\sqrt{1 + x}$.

7. $(a^{\frac{1}{3}} - x^{\frac{1}{3}})^{\frac{3}{2}}$.

8. $\dfrac{x - a}{a^2\sqrt{2\,ax - x^2}}$.

9. $(3\,x^4 - 2\,x^2 + 2)\sqrt{2\,x^2 + 1}$.

10. $\dfrac{(2\,ax - x^2)^{\frac{3}{2}}}{ax^3}$.

11. $\dfrac{\sqrt{x^2 + a^2}}{a^2x}$.

12. $(x^2 - 3)\sqrt[3]{x^2 + 1}$.

49. Differentiation of implicit functions. It frequently happens that the relation between two variables, as x and y, is conveniently expressed by means of an implicit function. In such cases the derivative $f'(x)$ can be found by differentiating according to the general formulas just given and finding the quotient $\dfrac{dy}{dx}$. Another method will be discussed in Chapter XIII. The expression for the derivative will generally contain both variables, and its numerical value can be found for known simultaneous values of x and y.

Ex. 1. Given $\qquad x^3y - 4\,xy^2 + 6\,x^2 - 3\,y = 0,$

we obtain by differentiation (using the formula for the product)

$$(3\,x^2y\,dx + x^3\,dy) - (4\,y^2\,dx + 8\,xy\,dy) + 12\,x\,dx - 3\,dy = 0,$$

whence
$$\frac{dy}{dx} = -\frac{3\,x^2y - 4\,y^2 + 12\,x}{x^3 - 8\,xy - 3}.$$

The values $x = 1$, $y = 1$ satisfy the original equation, and for these values the derivative has the value

$$\left.\frac{dy}{dx}\right]_{1,1} = \frac{11}{10} = 1.1;$$

that is, the slope of the curve is 1.1 at the point (1, 1).

Ex. 2. The equation representing the adiabatic expansion of air (*i.e.* expansion without gain or loss of heat) is $pv^k = C$, where k and C are constants. p denotes the pressure and v the volume of the air. Find the rate of change of pressure relative to the volume.

Differentiating, we obtain

$$v^k\,dp + kpv^{k-1}\,dv = 0,$$

or

$$v\,dp + kp\,dv = 0,$$

whence

$$\frac{dp}{dv} = -k\frac{p}{v}.$$

EXERCISES

Find the derivative $\frac{dy}{dx}$ for each of the following functions:

1. $x^3 - xy^2 - c^3 = 0.$

2. $b^2x^2 - a^2y^2 = a^2b^2.$

3. $x^2 + y^{\frac{2}{3}} = c.$

4. $x^4 - x^3y + x^2y^2 + 2\,xy^3 = 0.$

5. Find the general expression for the slope of the tangent to the conic section whose equation is

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0.$$

6. Find the slope of the circle $x^2 + y^2 = 100$ at the point $(-8, 6)$.

7. Find the slope of the ellipse $9\,x^2 + 14\,y^2 = 50$ at the point (2, 1); at the point $(-2, 1)$.

8. Find the slope of the curve

$$x^3y - 5\,x^2y^2 + 12\,y^4 = 0$$

at the point (2, 1).

9. Derive expressions for the polar subtangent and polar subnormal of the curves:

(*a*) $\rho^2 - \rho\theta^{\frac{1}{2}} = C.$

(*b*) $\rho - \rho^2\theta + C = 0.$

10. Derive general expressions for the derivative $\frac{dp}{dv}$ when the expansion of a gas follows the following laws, respectively:

(*a*) $p^mv^n = C$, (m, n, and C constants);

(*b*) $\left(p + \frac{a}{v^2}\right)(v - b) = C.$

Find $\frac{dy}{dx}$ in the following:

11. $(1+x)^2y=(1-x)y^2-x^3.$ **12.** $xy^2=b^2(b-x).$

13. $x\sqrt{y}-y\sqrt{x}=C.$

50. Applications of differentials. Suppose we have given

$$y=f(x),$$

and by differentiation we obtain

$$dy=f'(x)\,dx. \tag{1}$$

The differentials dy and dx may be taken as representing the derivatives D_ty and D_tx, where t denotes any third variable. If we consider the variable t as denoting time, then dy and dx represent the time rates of the function y and of the variable x respectively. Hence, if the time rate of x is given, the time rate of y is found by differentiation.

Ex. **1.** Boyle's law for the expansion of air is expressed by the equation $pv=C$. At a given instant the pressure is 40 lb./sq. in., the volume of the air is 8 cu. ft., and the volume is increasing at the rate of 0.5 cu. ft./sec. At what rate is the pressure changing?

We have $$pv=C,$$
whence $$p\,dv+v\,dp=0,$$
or $$dp=-\frac{p}{v}dv.$$
Since $$dv=D_tv=0.5,$$
$$dp=-\frac{40}{8}\times 0.5=-2.5.$$

Hence, the pressure is *decreasing* at the rate of 2.5 lb./sq. in. per second.

If in equation (1) the differentials are replaced by the increments Δy and Δx, the result is an *approximate* relation

$$\Delta y=f'(x)\,\Delta x, \tag{2}$$

which is frequently useful in finding the error in the result of a computation due to a small error in the observed data upon which the computation is based. The **relative error** $\frac{\Delta y}{y}$ is given approximately by the equation

$$\frac{\Delta y}{y}=\frac{f'(x)}{f(x)}\Delta x. \tag{3}$$

Ex. **2**. The area A of a circle being determined from a measurement of the diameter D, find the relative error in the calculated area due to an error in the measurement.

Since $$A = \tfrac{1}{4}\pi D^2,$$
we have $$dA = \tfrac{1}{2}\pi D \cdot dD,$$
whence approximately $$\Delta A = \tfrac{1}{2}\pi D \cdot \Delta D,$$
and $$\frac{\Delta A}{A} = \frac{\tfrac{1}{2}\pi D}{\tfrac{1}{4}\pi D^2}\Delta D = 2\frac{\Delta D}{D}.$$

Hence an error of one per cent in the measurement of the diameter gives approximately an error of two per cent in the calculated area.

EXERCISES

1. A point moves in the straight line $5x - 3y = 30$ in such a way that the Y-component of its velocity is 6. Find the X-component and the velocity in the path.

2. Find the X- and Y-components of the velocity of a point that moves in the line $3x + 4y = 12$ with a speed of 15 units.

3. Suppose that a straight wire rotates about one end with an angular speed of 3 rad./sec. and that a bead on this wire moves along it with a speed of 8 ft./sec. When the bead is 2 feet from the center of rotation, (*a*) what is the component of its velocity perpendicular to the wire? (*b*) what is its velocity in its path?

4. Find the polar equation of the curve described by the bead, Ex. 3, (*a*) when the angular speed ω of the wire is a times the linear speed of the bead along the wire; (*b*) when ω is constant and the speed of the bead on the wire varies inversely as the distance from the center of rotation.

5. If a soap bubble's diameter is increasing at the rate of $\frac{1}{4}$ in./sec. when it is 4 inches, at what rate is the inclosed volume increasing?

6. If a man 6 feet in height is walking at the rate of 3 mi./hr. away from a lamp-post 30 feet high, at what rate is the end of his shadow receding from the lamp-post?

7. The time of a complete oscillation of a pendulum of length L is given by the formula
$$t = 2\pi\sqrt{\frac{L}{g}}.$$
Find the rate of change of the time compared with that of the pendulum's length.

8. Find the error in the calculated time if the error in the measurement of the length of the pendulum is 0.5 per cent.

9. Find an expression for the relative error in the volume of a sphere calculated from a measurement of the diameter if there is an error in the measurement.

10. From the formula for kinetic energy $T = \frac{1}{2} mv^2$, show that a small change in v involves approximately twice as great a relative change in T.

11. An engine cylinder has a diameter of 12 inches. At what speed is the piston moving when steam is entering the cylinder at the rate of 18 cu. ft. / sec.?

MISCELLANEOUS EXERCISES

Differentiate the following functions, using differentials.

1. $\dfrac{2x^2+1}{3x^3}\sqrt{1-x^2}$.

2. $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{x}$.

3. $5x^{\frac{2}{3}}(x^2-4)^{\frac{1}{2}}(x+7)^{\frac{1}{3}}$.

4. $\dfrac{x\sqrt{a+x}}{\sqrt{a}-\sqrt{a-x}}$.

Find $\dfrac{dy}{dx}$ for the functions defined by the following equations.

5. $x^3 - 5ax^2y + 7y^3 = 0$.

6. $x^2 - y^2 = \dfrac{3xy}{\sqrt{1-y^2}}$.

7. $\dfrac{xy}{x+y} = c$.

8. $\dfrac{1}{\sqrt{x^2+y^2}} = k$.

9. Find expressions for the velocity components of a point moving in the parabola $y^2 = 20x$ with a speed of 12 in. / sec. Find the values for the point (5, 10).

10. A point moves in the straight line $6x - 8y = 12$ with a velocity of 5 ft. / sec. Find the components of the velocity along the X- and Y-axes, respectively.

11. A point moves in the circle $x^2 + y^2 = 36$ with a velocity of 8 ft. / sec. Find the X- and Y-components when the point is at $(5, \sqrt{11})$.

12. A point moves along the curve $\rho = \dfrac{4}{\sqrt{\theta}}$. When $\rho = 3$, the component of the point's velocity along the radius vector is 5 ft. / sec. Find (a) the component perpendicular to the radius vector, and (b) the velocity in the curve.

13. A crank pin moves in a circle 2 feet in diameter with a constant speed of 28 ft. / sec. When the crank makes an angle of 30° with the horizontal, what is (a) the vertical component of the crank pin's velocity? (b) the horizontal component?

14. The path of a projectile is the parabola

$$y = x \tan \alpha - \frac{gx^2}{2 v_0^2 \cos^2 \alpha};$$

and the horizontal velocity ($D_t x$) is $v_0 \cos \alpha$. Show that the velocity in the path is $\sqrt{v_0^2 - 2gy}$, and that the vertical acceleration is g.

15. If a point moves in the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ so that the X-component of the velocity is constant, find the acceleration in the direction of the Y-axis.

16. Find the equation of the path of a point which moves in such a way that the X-component of the velocity is constant while the Y-component is negative and varies directly as the time. Give a physical illustration.

17. Sand or grain, when poured from a height on a level surface, forms a cone with a circular base and a constant angle β at the vertex, dependent on the material. Let a denote the radius of the base of the cone at a given time, and suppose material is being added at the rate of C cu. ft./sec. At what rate is the radius increasing?

18. The velocity of a jet of liquid issuing from an orifice is given by the formula, $v = \sqrt{2\,gh}$, where h is the height of the liquid surface above the orifice. If $h = 100$ feet and is decreasing at the rate of 0.2 ft./sec., find the rate at which v is decreasing. Take $g = 32.2$.

19. Given $x = 3\,y^2 + 7\,y + 1$; find $D_x y$ without first obtaining y as an explicit function of x.

20. Air expands according to the adiabatic law $pv^{1.4} = C$. When the pressure is 40 lb./sq. in., the volume is 5 cubic feet and is increasing at the rate of 0.2 cu. ft./sec. Find the rate at which the pressure is changing.

21. The formula for the electrical resistance of a platinum wire is

$$R = R_0\,(1 + a\tau + b\tau^2),$$

where R_0, a, and b are constants, and τ denotes the temperature of the wire. Find the rate of increase of resistance at the temperature τ_1, if the temperature is rising at the rate of 0.1° per second.

22. Another formula for the variation of electrical resistance of a metal wire with the temperature is $R = R_0\,(1 - e\tau + f\tau^2)^{-1}$. Find the rate of change of the resistance compared with that of the temperature at temperature τ_1.

23. Given $(x^2 + y^2)^2 = a^2\,(x^2 - y^2)$. (*a*) For what values of x has the curve representing this function a turning point? (*b*) For what value of x does the tangent make an angle of 45° with the X-axis? (*c*) Write the equation of the normal at $x = \frac{a}{2}$.

24. A hemispherical bowl 20 inches in diameter has an orifice in the bottom through which the water contained in the bowl is flowing. If, when the surface of the water is 8 inches above the bottom, the rate of flow is 16 cu. in./sec., at what rate is the water level falling, assuming that the surface remains plane?

CHAPTER V

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

51. Transcendental functions. All functions that are not algebraic (Art. 30) are called transcendental. The transcendental functions include trigonometric, inverse trigonometric, exponential, and logarithmic functions. In a former chapter we considered the differentiation of algebraic functions; in this chapter we shall develop formulas for the differentiation of the more elementary transcendental functions, starting with the trigonometric functions.

In the derivation of the following formulas, we shall assume u to be a continuous function of x, and we shall find the derivatives with respect to x.

52. Differentiation of sin u.

Let $$y = \sin u.$$

Then $$y + \Delta y = \sin (u + \Delta u)$$

and $$\Delta y = \sin (u + \Delta u) - \sin u$$

$$= \sin u \cos \Delta u + \sin \Delta u \cos u - \sin u.$$

Therefore, we have

$$\frac{\Delta y}{\Delta u} = \left[- \frac{\sin u\,(1 - \cos \Delta u)}{\Delta u} + \cos u \frac{\sin \Delta u}{\Delta u} \right],$$

and

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = \left[- \sin u \underset{\Delta u \doteq 0}{L} \frac{1 - \cos \Delta u}{\Delta u} + \cos u \underset{\Delta u \doteq 0}{L} \frac{\sin \Delta u}{\Delta u} \right].$$

The two limits in the second member have been evaluated; thus,

$$\underset{\Delta u \doteq 0}{L} \frac{1 - \cos \Delta u}{\Delta u} = 0, \text{ and } \underset{\Delta u \doteq 0}{L} \frac{\sin \Delta u}{\Delta u} = 1. \quad \text{(Arts. 13, 15, Ex. 5)}$$

Hence, $$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = D_u y = \cos u.$$

Since u is a function of x, we have by Art. 31

$$D_x y = D_u y \cdot D_x u.$$

Therefore $$D_x y = \cos u\ D_x u;$$

that is, $$\boldsymbol{D_x (\sin u) = \cos u\ D_x u.} \tag{1}$$

In the differential notation, we have

$$dy = \cos u\ du,$$

or $$\boldsymbol{d (\sin u) = \cos u\ du.} \tag{2}$$

In the special case where $u = x$, (1) becomes

$$\boldsymbol{D_x \sin x = \cos x.} \tag{3}$$

The other trigonometric functions may now be differentiated by applying the general laws of differentiation.

53. Differentiation of cos u.

Let $$y = \cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Then $$D_x y = \cos\left(\frac{\pi}{2} - u\right) D_x\left(\frac{\pi}{2} - u\right)$$

$$= -\sin u\ D_x u;$$

that is, $$\boldsymbol{D_x (\cos u) = -\sin u\ D_x u,} \tag{1}$$

or, in the notation of differentials,

$$\boldsymbol{d (\cos u) = -\sin u\ du.} \tag{2}$$

For $u = x$,

$$\boldsymbol{D_x (\cos x) = -\sin x.} \tag{3}$$

54. Differentiation of tan u.

If $$y = \tan u = \frac{\sin u}{\cos u},$$

then $$D_x y = \frac{\cos u\ D_x \sin u - \sin u\ D_x \cos u}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} D_x u$$

$$= \frac{1}{\cos^2 u} D_x u = \sec^2 u\ D_x u.$$

Hence $$\boldsymbol{D_x(\tan u) = \sec^2 u\, D_x u,} \tag{1}$$

or $$\boldsymbol{d(\tan u) = \sec^2 u\, du.} \tag{2}$$

Also $$\boldsymbol{D_x(\tan x) = \sec^2 x.} \tag{3}$$

55. Differentiation of cot u.

Let $$y = \cot u = \frac{\cos u}{\sin u};$$

then $$D_x y = \frac{\sin u\, D_x \cos u - \cos u\, D_x \sin u}{\sin^2 u}$$

$$= -\frac{\sin^2 u + \cos^2 u}{\sin^2 u} D_x u = -\csc^2 u\, D_x u.$$

Hence $$\boldsymbol{D_x(\cot u) = -\csc^2 u\, D_x u,} \tag{1}$$

or $$\boldsymbol{d(\cot u) = -\csc^2 u\, du.} \tag{2}$$

For $u = x$, $$\boldsymbol{D_x(\cot x) = -\csc^2 x.} \tag{3}$$

56. Differentiation of sec u and csc u.

If $$y = \sec u = \frac{1}{\cos u},$$

we have $$D_x y = -\frac{1}{\cos^2 u} D_x(\cos u)$$

$$= \sec^2 u \sin u\, D_x u = \sec u \tan u\, D_x u.$$

Therefore $$\boldsymbol{D_x(\sec u) = \sec u \tan u\, D_x u,} \tag{1}$$

and $$\boldsymbol{d(\sec u) = \sec u \tan u\, du.} \tag{2}$$

Proceeding as above, we find

$$\boldsymbol{D_x(\csc u) = -\csc u \cot u\, D_x u,} \tag{3}$$

or $$\boldsymbol{d(\csc u) = -\csc u \cot u\, du.} \tag{4}$$

The details of the proof are left to the student.

Ex. Differentiate $f(x) = \tan^2 \sqrt{a^2 - x^2}$.

We have $$y = \tan^2 \sqrt{a^2 - x^2},$$

whence $$D_x y = 2 \tan \sqrt{a^2 - x^2}\, D_x \tan \sqrt{a^2 - x^2}.$$

From (1) Art. 54, taking $u = \sqrt{a^2 - x^2}$, we obtain

$$D_x \tan \sqrt{a^2 - x^2} = \sec^2 \sqrt{a^2 - x^2}\, D_x \sqrt{a^2 - x^2}$$

$$= - \sec^2 \sqrt{a^2 - x^2} \frac{x}{\sqrt{a^2 - x^2}}.$$

Hence,
$$f'(x) = - \frac{2x}{\sqrt{a^2 - x^2}} \tan \sqrt{a^2 - x^2} \sec^2 \sqrt{a^2 - x^2}.$$

EXERCISES

Differentiate the following.

1. $y = \cos ax$.

2. $y = \tan^3 x$.

3. $y = \sin 2x - \cos 2x$.

4. $y = \sec^2 x$.

5. $y = \sin^2 3x$.

6. $y = \cos^3 (a^2 - x^2)$.

7. $y = x \sin x$.

8. $y = x^2 \tan \frac{x}{2}$.

9. $y = \cos \sqrt{x^2 - a^2}$.

10. $y = x - \tan x$.

11. $y = x^3 \csc 2x$.

12. $y = \sin x - x \cos x$.

13. $y = 2x \cos x + (x^2 - 2) \sin x$.

14. $y = x - \sin x \cos x$.

15. $y = \sin x (\cos^2 x + 2)$.

16. $\rho = 2a \frac{\sin^2 \theta}{\cos \theta}$.

17. $\rho = a \sqrt{\cos 2\theta}$.

18. $\rho = a (\sin n\theta)^{\frac{1}{n}}$.

19. $\rho = a \sec^3 \frac{\theta}{3}$.

20. $\rho = \frac{1}{1 - \cos \theta}$.

21. $s = A \cos \omega t - B \sin \omega t$.

22. $s = - r \cos \theta + L \left(1 - \sqrt{1 - \frac{r^2}{L^2} \sin^2 \theta}\right)$.

23. Find the derivative $D_x y$, when

$$x = a (\phi - \sin \phi),$$
$$y = a (1 - \cos \phi).$$

24. Let
$$x = a \cos \phi + a\phi \sin \phi,$$
$$y = a \sin \phi - a\phi \cos \phi.$$

Find $D_\phi x$, $D_\phi y$, and $D_x y$.

25. Find the polar subtangent and polar subnormal of the curves:

(*a*) $\rho = \sin \theta$. (*b*) $\rho = a (1 - \cos \theta)$. (*c*) $\rho = a \sec^2 \frac{\theta}{2}$.

26. From the equation $\sin 2\theta = 2 \sin \theta \cos \theta$, derive by differentiation a formula for $\cos 2\theta$.

27. Find expressions for the subtangent and subnormal of the curve $y = a \sin x$.

28. Find the angle at which the curves $y = \cos x$ and $y = \tan x$ intersect.

29. Find the angle which the curves $y = \sin x$ and $y = \cos x$ make with each other at their points of intersection.

30. Find the value of θ for which $\tan \theta$ is increasing twice as fast as θ.

31. When $\theta = 22°$, find approximately the changes in $\sin \theta$ and $\cos \theta$, for a change of $1'$ in the angle.

32. In a triangle two sides a and b include an angle θ. If the sides remain of constant length and the angle is varied, find the rate of change of the area of the triangle with respect to θ when $\theta = \frac{\pi}{4}$.

33. In a certain type of motion the velocity is given by the expression $v = v_0 \cos kt$. Find an expression for the acceleration.

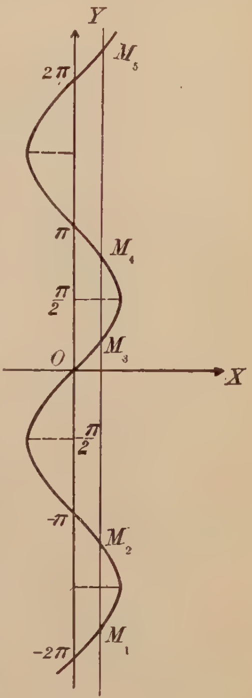

FIG. 22.

57. Inverse trigonometric functions. The trigonometric functions are all **single-valued** functions; that is, for each value of the variable there corresponds one and only one value of the function. Thus, from $x = \sin y$, x has but one value for each value of y. The inverse trigonometric functions are not single-valued, but **multiple-valued** functions, for to each value of variable there corresponds any number of values of the function. In Fig. 22 is given the graph of $y = \text{arc} \sin x$. For $x = a$, it will be seen that y may have any one of the *values* indicated by the points $M_1, M_2, M_3, \cdots$. If, however, the values of y be restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then within this range the function is single-valued.

The graph of $y = \text{arc} \sec x$ is given in Fig. 23. From the examination of this graph, it will be seen that the function will be single-valued if $0 \leqq y \leqq \pi$.

Ex. Plot roughly the graphs of $\text{arc} \cos x$, $\text{arc} \tan x$, and $\text{arc} \cot x$, and determine limits within which these functions are single-valued.

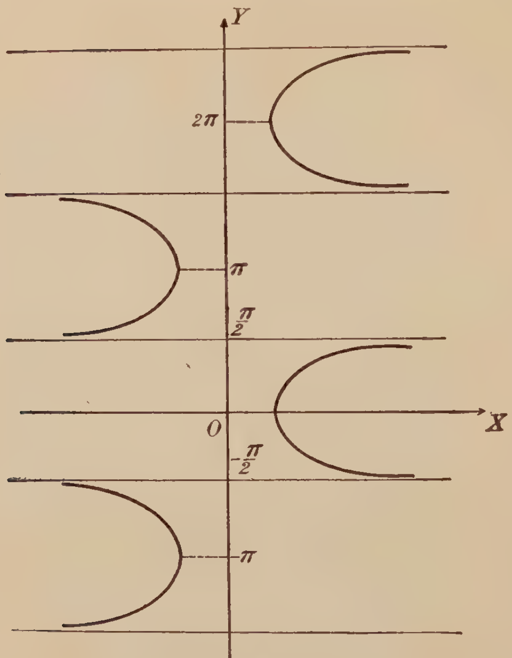

FIG. 23.

58. Differentiation of arc sin *u* and arc cos *u*.

Let $$y = \text{arc} \sin u,$$

then $$u = \sin y.$$

Differentiating, we obtain $$D_y u = \cos y.$$

Therefore $$D_u y = \frac{1}{\cos y},$$

and $$D_x y = D_u y \cdot D_x u \qquad \text{(Art. 31)}$$

$$= \frac{1}{\cos y} D_x u.$$

Since $$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - u^2},$$

we have $$D_x y = \frac{1}{\sqrt{1 - u^2}} D_x u.$$

Hence, $$D_x \text{ arc sin } u = \frac{1}{\sqrt{1-u^2}} D_x u, \tag{1}$$

or in the differential notation,

$$d \text{ arc sin } u = \frac{du}{\sqrt{1-u^2}}. \tag{2}$$

If, as a special case, $u = x$,

$$D_x \text{ arc sin } x = \frac{1}{\sqrt{1-x^2}}. \tag{3}$$

Proceeding in the same way with $y = \text{arc cos } u$, we obtain

$$D_x \text{ arc cos } u = -\frac{1}{\sqrt{1-u^2}} D_x u, \tag{4}$$

$$d \text{ arc cos } u = -\frac{du}{\sqrt{1-u^2}}, \tag{5}$$

$$D_x \text{ arc cos } x = -\frac{1}{\sqrt{1-x^2}}. \tag{6}$$

Since the sign of the radical $\sqrt{1-x^2}$ may be either positive or negative, the signs of the derivatives in (3) and (6) are ambiguous. Reference to the graph of arc sin x, Fig. 22, shows how the ambiguity arises. Thus for $x = a$, the derivative is positive at M_1, M_3, M_5, etc., and negative at points M_2, M_4, etc. When y is restricted to the interval $-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$ so as to make the function single-valued, the radical must be taken with the positive sign. The student may show that if $0 \leqq y \leqq \pi$, the function $y = \text{arc cos } x$ is single-valued and the derivative given by (6) has the proper sign when the radical is given the positive sign.

59. Differentiation of arc tan u and arc cot u.

Let $$y = \text{arc tan } u,$$

whence $$u = \tan y,$$

and $$D_y u = \sec^2 y = 1 + u^2.$$

We have then $$D_u y = \frac{1}{\sec^2 y} = \frac{1}{1+u^2}.$$

But by Art. 31, $$D_x y = D_u y \cdot D_x u = \frac{1}{1+u^2} D_x u;$$

therefore, $$\boldsymbol{D_x \text{ arc tan } u = \frac{1}{1+u^2} D_x u,} \tag{1}$$

and $$\boldsymbol{d \text{ arc tan } u = \frac{du}{1+u^2}.} \tag{2}$$

For $u = x$ $$\boldsymbol{D_x \text{ arc tan } x = \frac{1}{1+x^2}.} \tag{3}$$

To differentiate arc cot u, we may proceed as above, or we may derive the result from (1). Thus, since

$$y = \text{arc cot } u = \text{arc tan } \frac{1}{u},$$

we have

$$D_x y = \frac{1}{1+\left(\frac{1}{u}\right)^2} D_x\left(\frac{1}{u}\right) = \frac{1}{1+\left(\frac{1}{u}\right)^2} \cdot \left(-\frac{1}{u^2}\right) D_x u = -\frac{1}{1+u^2} D_x u.$$

Hence, we have $$\boldsymbol{D_x \text{ arc cot } u = -\frac{1}{1+u^2} D_x u,} \tag{4}$$

$$\boldsymbol{d \text{ arc cot } u = -\frac{du}{1+u^2},} \tag{5}$$

$$\boldsymbol{D_x \text{ arc cot } x = -\frac{1}{1+x^2}.} \tag{6}$$

60. Differentiation of arc sec u and arc csc u.

Let $$y = \text{arc sec } u,$$

then $$u = \sec y,$$

and $$D_y u = \sec y \tan y.$$

Hence, we have $$D_u y = \frac{1}{\sec y \tan y},$$

and $$D_x y = D_u y \, D_x u \qquad \textbf{(Art. 31)}$$

$$= \frac{1}{\sec y \tan y} D_x u.$$

Since $\sec y = u$, and $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{u^2 - 1}$,

we may write $$D_x y = \frac{1}{u\sqrt{u^2-1}} D_x u.$$

Therefore $$D_x \operatorname{arc\,sec} u = \frac{1}{u\sqrt{u^2-1}} D_x u, \tag{1}$$

and $$d \operatorname{arc\,sec} u = \frac{du}{u\sqrt{u^2-1}}. \tag{2}$$

Also $$D_x \operatorname{arc\,sec} x = \frac{1}{x\sqrt{x^2-1}}. \tag{3}$$

Proceeding in the same way, we obtain

$$D_x \operatorname{arc\,csc} u = -\frac{1}{u\sqrt{u^2-1}} D_x u, \tag{4}$$

$$d \operatorname{arc\,csc} u = -\frac{du}{u\sqrt{u^2-1}}, \tag{5}$$

$$D_x \operatorname{arc\,csc} x = -\frac{1}{x\sqrt{x^2-1}}. \tag{6}$$

From the graph of arc sec x, Fig. 23, it is readily seen that for $0 \leqq y \leqq \frac{\pi}{2}$, the derivative is positive and therefore the positive sign of the radical $\sqrt{x^2-1}$ must be taken; but for $\frac{\pi}{2} \leqq y \leqq \pi$ the derivative is positive while x is negative, and therefore for this interval the negative sign of the radical must be taken. From the graph for arc csc x the student may deduce a similar statement relative to the radical in (6).

Ex. $y = \arctan \dfrac{x}{\sqrt{1-2x^2}}$.

Let $\dfrac{x}{\sqrt{1-2x^2}} = u$; then $y = \arctan u$, $dy = \dfrac{du}{1+u^2}$,

and $$du = d\frac{x}{\sqrt{1-2x^2}} = \frac{\sqrt{1-2x^2} + \dfrac{2x^2}{\sqrt{1-2x^2}}}{1-2x^2}dx = \frac{1}{(1-2x^2)^{\frac{3}{2}}}dx.$$

Substituting these values of u and du, we get

$$dy = \frac{1}{1+\dfrac{x^2}{1-2x^2}} \cdot \frac{1}{(1-2x^2)^{\frac{3}{2}}}dx = \frac{1}{(1-x^2)\sqrt{1-2x^2}}dx.$$

EXERCISES

Differentiate the following:

1. $y = \text{arc}\cos\frac{x}{a}$.
2. $y = \text{arc}\tan\frac{x}{\sqrt{k}}$.
3. $y = \text{arc}\sec\frac{x^2}{a^2}$.
4. $y = \text{arc}\sin\sqrt{a^2 - x^2}$.
5. $y = x\,\text{arc}\tan x$.
6. $\theta = \text{arc}\sec\frac{\rho}{\sqrt{\rho^2 - a^2}}$.
7. $\theta = \text{arc}\sin\frac{\sqrt{\rho^2 - a^2}}{\rho}$.
8. $y = \text{arc}\tan\frac{x}{a} + \text{arc}\tan\frac{a}{x}$.
9. $y = \text{arc}\cos(\cos x)$.
10. $y = \text{arc}\tan\sqrt{1 - k^2\tan^2 x}$.
11. $y = x^2\,\text{arc}\cos x^2$.
12. $y = \frac{1}{mn}\text{arc}\tan\frac{x}{m}$.
13. $y = \frac{\sqrt{a^2 - x^2}}{x} + \text{arc}\sin\frac{x}{a}$.
14. $y = \sqrt{x^2 - a^2} - a\,\text{arc}\sec\frac{x}{a}$.
15. $y = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\text{arc}\sin\frac{x}{a}$.
16. $y = x\,\text{arc}\cos x - \sqrt{1 - x^2}$.

61. Exponential and logarithmic functions. The differentiation of the exponential functions a^u and e^u depends upon the evaluation of the limit

$$\underset{x \doteq 0}{L}\frac{a^x - 1}{x},$$

and this limit in turn involves the limit $\underset{x = \infty}{L}\left(1 + \frac{1}{x}\right)^x$. It will be necessary to consider these two limits before the desired formulas for differentiation can be deduced.

$\underset{x = \infty}{L}\left(1 + \frac{1}{x}\right)^x$. It can readily be shown that $\underset{n = \infty}{L}\left(1 + \frac{1}{n}\right)^n$, where n is a positive integer, is some number lying between 2 and 4. The outline of the proof is as follows: Take two numbers a and b such that $a > b > 0$ and let n be a positive integer. Then we have the identity,

$$\frac{a^{n+1} - b^{n+1}}{a - b} = a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + b^n, \tag{1}$$

from which follows the inequality

$$\frac{a^{n+1} - b^{n+1}}{a - b} < a^n(n + 1),$$

or

$$a^{n+1} - b^{n+1} < a^n(a - b)(n + 1).$$

The last inequality may be thrown into the form,

$$a^n[a-(a-b)(n+1)]<b^{n+1}. \tag{2}$$

Now choose for a and b two sets of numbers, subject to the condition $a>b>0$. Let these be:

$$(a)\quad a=1+\frac{1}{n},\ b=1+\frac{1}{n+1},$$

$$(b)\quad a=1+\frac{1}{2n},\ b=1.$$

Substituting these values respectively in (2), we get the two inequalities

$$\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}, \tag{3}$$

$$\frac{1}{2}\left(1+\frac{1}{2n}\right)^n<1. \tag{4}$$

From (3) it appears that the function $\left(1+\frac{1}{n}\right)^n$ increases with n; that is, the function is monotone. For $n=1$, the function takes the value 2. From the inequality (4) we obtain, by squaring both members,

$$\left(1+\frac{1}{2n}\right)^{2n}<4,$$

whence it follows that the function $\left(1+\frac{1}{n}\right)^n$ cannot exceed 4.

The limit required must therefore exist (Art. 14) and it lies between 2 and 4. The exact value of this limit is 2.7182818285 ⋯, as will be shown hereafter in connection with infinite series. This number is denoted by e and is the base of the natural system of logarithms.

In the preceding discussion n was given only positive integral values. The limit is the same, however, if instead of n we take a variable x which can take all real values. Hence, in general, we have

$$\underset{x=\infty}{L}\left(1+\frac{1}{x}\right)^x=e. \tag{5}$$

$\underset{x\doteq 0}{L}\dfrac{a^x-1}{x}$. To obtain the limit of $\dfrac{a^x-1}{x}$ as x approaches zero, let $y=a^x-1$. Then $x=\log_a(1+y)$ and as $x\doteq 0$, $y\doteq 0$. We have therefore

$$\underset{x\doteq 0}{L}\frac{a^x-1}{x}=\underset{y\doteq 0}{L}\frac{y}{\log_a(1+y)}=\underset{y\doteq 0}{L}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}$$

$$=\frac{1}{\log_a \underset{y\doteq 0}{L}(1+y)^{\frac{1}{y}}}=\frac{1}{\log_a e}.$$

Since, however, $\frac{1}{\log_a e} = \log a$, we have finally

$$\underset{x \doteq 0}{L} \frac{a^x - 1}{x} = \log a. \tag{6}$$

The substitution of e for a in (6) gives

$$\underset{x \doteq 0}{L} \frac{e^x - 1}{x} = \log e = 1. \tag{7}$$

62. Differentiation of a^u and e^u.

Let $$y = a^u;$$

then $$y + \Delta y = a^{u+\Delta u},$$

and $$\Delta y = a^{u+\Delta u} - a^u = a^u(a^{\Delta u} - 1).$$

Therefore $$\frac{\Delta y}{\Delta u} = a^u \frac{a^{\Delta u} - 1}{\Delta u}.$$

Taking limits, we have

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = a^u \underset{\Delta u \doteq 0}{L} \frac{a^{\Delta u} - 1}{\Delta u}.$$

But, as shown in the preceding discussion,

$$\underset{\Delta u \doteq 0}{L} \frac{a^{\Delta u} - 1}{\Delta u} = \log a;$$

hence, we have $$D_u y = \underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = a^u \log a.$$

Using the theorem of Art. 31, we have

$$D_x y = D_u y \cdot D_x u$$
$$= a^u \log a \, D_x u;$$

that is, $$D_x a^u = a^u \log a \, D_x u, \tag{1}$$

or $$da^u = a^u \log a \, du. \tag{2}$$

If $u = x$, we have

$$D_x a^x = a^x \log a, \quad \text{or} \quad da^x = a^x \log a \, dx. \tag{3}$$

If in (1) we substitute e for a, we obtain (since $\log e = 1$)

$$D_x e^u = e^u D_x u; \quad de^u = e^u du. \tag{4}$$

For $u = x$, we have

$$D_x e^x = e^x; \qquad de^x = e^x\,dx. \tag{5}$$

It may be noted that e^x is a function whose derivative is the function itself.

63. Differentiation of $\log_a u$.

Let $\qquad y = \log_a u,$

whence $\qquad u = a^y.$

From the preceding article, we obtain

$$D_y u = a^y \log a,$$

whence
$$D_u y = \frac{1}{a^y \log a}.$$

By the theorem of Art. 31,

$$D_x y = D_u y \cdot D_x u$$

$$= \frac{1}{a^y \log a} D_x u = \frac{1}{u \log a} D_x u.$$

We have therefore

$$D_x \log_a u = \frac{1}{\log a}\frac{D_x u}{u}, \text{ or } d \log_a u = \frac{1}{\log a}\frac{du}{u}. \tag{1}$$

If $u = x$, we have

$$D_x \log_a x = \frac{1}{\log a}\frac{1}{x}; \qquad d \log_a x = \frac{1}{\log a}\frac{dx}{x}. \tag{2}$$

The fraction $\dfrac{1}{\log a}$ is called the **modulus** of the system of logarithms whose base is a, and may be denoted by m. For the Briggs' or common system, in which $a = 10$, $m = .434294 \cdots$.

We have, therefore,
$$D_x \log_a x = \frac{m}{x}.$$

64. Differentiation of $\log u$.

By the substitution of e for a in (1) and (2) of the preceding article, the following formulas are obtained (since $\log e = 1$):

$$D_x \log u = \frac{1}{u} D_x u; \quad d \log u = \frac{du}{u}. \tag{1}$$

$$D_x \log x = \frac{1}{x}; \qquad d \log x = \frac{dx}{x}. \tag{2}$$

Ex. 1. $$y = \log \sin x.$$

$$dy = \frac{1}{\sin x}\, d \sin x = \frac{1}{\sin x} \cos x \, dx = \cot x \, dx.$$

Ex. 2. $$y = e^{\sqrt{1+x^2}}.$$

$$dy = e^{\sqrt{1+x^2}}\, d(\sqrt{1+x^2}) = e^{\sqrt{1+x^2}} \frac{x}{\sqrt{1+x^2}}\, dx.$$

65. Logarithmic differentiation. If a function consists of a number of factors, it may be conveniently differentiated by taking logarithms. Thus, if

$$y = uvw,$$

$$\log y = \log u + \log v + \log w.$$

Differentiating, we have

$$\frac{dy}{y} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w},$$

or
$$dy = \frac{y}{u}\, du + \frac{y}{v}\, dv + \frac{y}{w}\, dw.$$

Ex. 1. Differentiate the function $y = \dfrac{x\sqrt{x^2-3}}{(x+7)^2}$.

Taking logarithms, we have

$$\log y = \log x + \tfrac{1}{2} \log (x^2 - 3) - 2 \log (x + 7),$$

whence by differentiation

$$\frac{dy}{y} = \left(\frac{1}{x} + \frac{x}{x^2-3} - \frac{2}{x+7}\right) dx$$

$$= \frac{14x^2 + 3x - 21}{x(x+7)(x^2-3)}\, dx.$$

Therefore

$$\frac{dy}{dx} = \frac{x\sqrt{x^2-3}}{(x+7)^2} \cdot \frac{14x^2 + 3x - 21}{x(x+7)(x^2-3)}$$

$$= \frac{14x^2 + 3x - 21}{(x+7)^3\sqrt{x^2-3}}.$$

Logarithmic differentiation is especially useful in differentiating an exponential function having a variable base.

EX. 2. Given
$$y = e^x x^{\frac{1}{x}},$$
we have
$$\log y = \log e^x + \frac{1}{x}\log x$$
$$= x + \frac{1}{x}\log x.$$
Differentiating, we have
$$\frac{dy}{y} = dx - \frac{1}{x^2}\log x\,dx + \frac{1}{x^2}\,dx,$$
$$\frac{dy}{dx} = y\left[1 + \frac{1}{x^2}(1 - \log x)\right]$$
$$= e^x x^{\frac{1}{x}} + e^x x^{\frac{1}{x}-2}(1 - \log x).$$

Logarithmic differentiation also enables us to differentiate u^n, where n may have any constant value. We have
$$y = u^n,$$
whence
$$\log y = n \log u.$$
Differentiating, we obtain
$$\frac{1}{y}D_x y = \frac{n}{u}D_x u,$$
whence
$$D_x y = n\frac{y}{u}D_x u.$$
or
$$\boldsymbol{D_x y = n u^{n-1} D_x u.}$$

This completes the demonstration begun in Art. 29, and shows that the formula given there for the derivative of u^n holds for all *real* values of n.

EXERCISES

Differentiate the following:

1. $y = \log(x^2 - 3x + 5)$.
2. $y = \log_a\left(\dfrac{x+m}{x-m}\right)$.
3. $y = a^{\sqrt{x^2 + a^2}}$.
4. $y = e^{x^3} + e^x$.
5. $y = x \log x$.
6. $y = \log(\log x)$.
7. $y = e^x + e^{-x}$.
8. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.
9. $y = e^{\sin x}$.
10. $y = \log \tan x$.

11. $y = \log \dfrac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$.

12. $y = \log(x + \sqrt{x^2 - a^2})$.

13. $y = \dfrac{1}{a} \log \dfrac{x}{a + \sqrt{x^2 + a^2}}$.

14. $y = x \arctan x - \log \sqrt{1 + x^2}$.

15. $y = \dfrac{e^{ax}}{a^2}(ax - 1)$.

16. $y = \dfrac{1}{2b} \log \dfrac{a + bx^2}{b}$.

In Exs. 17 to 20, differentiate by taking logarithms:

17. $y = x(1 - x)\sqrt{1 + x^2}$.

18. $y = \dfrac{x^2\sqrt{x^2 - 4}}{(x+3)^2}$.

19. $y = \dfrac{3x^2 - 1}{x\sqrt{1 - x^2}}$.

20. $y = \sqrt{\dfrac{1 - x}{1 + x}}$.

21. If $z = Ae^{\theta\sqrt{k}} + Be^{-\theta\sqrt{k}}$, find $\dfrac{dz}{d\theta}$.

22. Find an expression for the velocity when the distance traversed by a moving point is expressed as a function of the time by the equation

$$s = (A + Bt)e^{-kt}.$$

23. If $s = e^{-kt}[a \sin mt + b \cos mt]$, find an expression for the velocity.

24. Find the polar subtangent, polar subnormal, lengths of polar tangent and polar normal of the curve $\rho = e^{a\theta}$.

25. Find the angle between the curve $y = \log x$ and the axis of x; between the same curve and the line $y = 2$.

26. Find the subtangent, subnormal, and the length of the normal of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

27. Given $\log 4.32 = 1.4633$, find approximately the value of $\log 4.33$ by means of the theorem $\Delta y = f'(x)\Delta x$.

28. By logarithmic differentiation derive the result $da^u = a^u \log a\, du$.

MISCELLANEOUS EXERCISES

1. Derive the formulas for the derivatives of $\cos u$, $\tan u$, and $\sec u$ directly from the defining equation

$$\underset{\Delta u \doteq 0}{L} \frac{\Delta y}{\Delta u} = \underset{\Delta u \doteq 0}{L} \frac{f(u + \Delta u) - f(u)}{\Delta u}$$

and the theorem of Art. 31.

2. Obtain the derivative of $\cot u$ from the relation $\cot u = \dfrac{1}{\tan u}$; also the derivative of $\sec u$ from $\sec u = \sqrt{1 + \tan^2 u}$.

3. Write out formulas for the derivatives of the inverse circular functions for the case in which $u = \frac{x}{a}$.

4. Discuss the following curves by means of their derivatives. Find where they cross the X-axis, the angles at which they cross, and the point at which their tangents are parallel to the X-axis.

(*a*) $y = \sin\left(x - \frac{\pi}{4}\right)$. (*b*) $y = \sin 2x$.

(*c*) $y = \sin x + \cos x$. (*d*) $y = \log x$.

5. When $\theta = 36°$, what is the rate of increase of (*a*) $\sin\theta$; (*b*) $\cos\theta$; (*c*) $\tan\theta$ compared with the rate of increase of θ?

6. The cycloid is given by the equations

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Derive an expression for the slope of the tangent, and find the value of this slope when $a = 3$, $\theta = \frac{2}{3}\pi$. For what value of θ is the tangent parallel to the X-axis?

Differentiate the following:

7. $\sqrt{a^2 - x^2} - a \log \dfrac{a + \sqrt{a^2 - x^2}}{x}$.

8. $\dfrac{x - 3a}{2}\sqrt{2ax + x^2} + \dfrac{3}{2}a^2 \log\left(x + a + \sqrt{2ax + x^2}\right)$.

9. $\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\arcsin\dfrac{x}{a}$.

10. $\dfrac{1}{x} - \dfrac{1}{3x^3} + \arctan x$. **11.** $\log(e^x + e^{-x})$. **12.** $\tan a^{\frac{1}{x}}$.

13. Find the polar subtangents and subnormals of the following curves:

(*a*) $\rho = a(1 + \cos\theta)$. (*b*) $\rho^2 = e^{a\theta}$.

(*c*) $\rho = a^2 \sin^2\theta$. (*d*) $\rho = \sec^2\dfrac{\theta}{2}$.

14. Find expressions for $\tan\psi$ (Art. 40) for the curves of Ex. 13.

15. If a point moves in a logarithmic spiral $\rho = e^{a\theta}$, show that the components of its velocity along and perpendicular to the radius vector have a constant ratio.

16. A point moves in the curve $\rho = a(1 - \cos\theta)$ with a speed of m units. Find the velocity components along and perpendicular to the radius vector.

17. The following formulas give approximately the relation between the pressure and temperature of saturated steam :

$$(a)\quad \log p = A + \frac{B}{T} + \frac{C}{T^2}. \qquad (T = \tau + 461)$$

$$(b)\quad \log p = m - n \log \frac{T}{T - C}.$$

$$(c)\quad \log p = a + b\alpha^{\theta} - c\beta^{\theta}. \qquad (\theta = \tau - 32)$$

From each formula derive an expression for the rate of change of the pressure with the temperature.

18. A point moves in accordance with the law expressed by the equation

$$s = ae^{-\lambda t} \cos 2\pi(bt + c).$$

Derive an expression for the velocity.

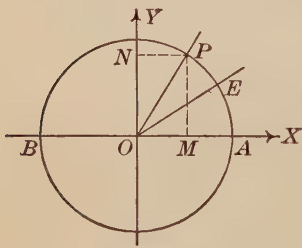

FIG. 24.

19. A radius OP of length a, Fig. 24, rotates with constant angular speed ω about the point O, and the projection M of P on the X-axis therefore moves on this axis. Show that $OM = x = a \cos \omega t$, and derive expressions for the velocity and acceleration of the point M.

(The motion of M is called *simple harmonic motion.*)

20. Taking the time t as abscissa, draw curves showing the displacement x, velocity v, and acceleration a of the point M. Take $t = 0$ when M is at A.

CHAPTER VI

INTEGRATION

66. Anti-derivatives and integrals. Thus far we have been chiefly concerned in finding the derivatives of given functions. We shall now consider the inverse operation; that is, having given a function $\phi(x)$, to find another function $f(x)$ such that $D_x f(x) = \phi(x)$. This inverse operation is called **integration**, and the resulting function is called the **anti-derivative** or **integral** of the given function. The function integrated is called the **integrand**.

Two symbols are in use to denote the inverse operation of integration. The symbol D_x^{-1} may be prefixed to the integrand; thus, since

$$D_x x^3 = 3\,x^2,$$

we have

$$D_x^{-1}\,3\,x^2 = x^3.$$

It is the universal custom, however, to denote integration by placing the symbol $\int$ before the *differential*. Thus, since

$$d(x^3) = 3\,x^2\,dx,$$

we write

$$\int 3\,x^2\,dx = x^3.$$

The direct and inverse operations may therefore be indicated by the symbols D_x and D_x^{-1}; or by the symbols d and $\int$.

EXERCISES

Find anti-derivatives of the following functions:

1. $5\,x^4$. **2.** $2\,x$. **3.** $\cos x$. **4.** $\dfrac{1}{x}$. **5.** $-\dfrac{1}{x^2}$.

Perform the following integrations:

6. $\int \sec^2\theta\,d\theta$. **7.** $\int 2\,at\,dt$. **8.** $\int \dfrac{dx}{1+x^2}$. **9.** $\int x^3\,dx$.

67. General theorems. It will be observed that the functions x^3 and $x^3 + C$, where C is any constant, have the same differential $3\,x^2\,dx$. Hence the integral $\int 3\,x^2\,dx$ should have the general form $x^3 + C$. In general, since the derivative of a function plus a constant is always equal to the derivative of the function, we have the following:

THEOREM I. *The general form of the anti-derivative or integral of a function must involve an additive constant. This constant is arbitrary, and is called the* **constant of integration.**

The value of the constant of integration in any particular case may be determined when certain initial conditions in the problem under discussion are known. The determination of this constant from such conditions will be subsequently discussed more fully.

An anti-derivative or integral function may be represented by a graph. The geometrical significance of annexing the constant of integration is to increase or decrease each ordinate of this graph by the same length. In other words, as we give various values to the constant of integration, we move the integral curve up or down. See Fig. 8.

It has been shown that the derivative of the product of a constant and a function is the product of the constant and the derivative of the function; that is, $D_x\, c \cdot f(x) = c \cdot D_x f(x)$ or in the differential notation, $d(cu) = c\,du$. The inverse operation gives a similar theorem, namely:

THEOREM II. *A constant factor in the integrand may be written either before or after the symbol* $\int$; *that is*

$$\int c\,du = c\int du. \tag{1}$$

The derivative of the sum of a finite number of functions was obtained by adding the derivatives of the separate functions. The inverse operation gives, therefore, the following theorem:

THEOREM III. *The integral of the algebraic sum of a finite number of functions is equal to the algebraic sum of the integrals of these functions; that is,*

$$\int (du + dv + dw) = \int du + \int dv + \int dw. \tag{2}$$

68. The integral $\int u^n\,du$. By differentiating the function u^{n+1}, we obtain

$$d(u^{n+1}) = (n+1)u^n\,du,$$

whence

$$d\left(\frac{u^{n+1}}{n+1}\right) = u^n\,du.$$

Therefore, we have

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$$

which holds for all values of n except the value -1. This formula is especially useful in the integration of integral algebraic functions.

Ex. 1. $\int 5x^3\,dx = 5\int x^3\,dx = \frac{5}{4}x^4 + C.$

Ex. 2. $\int (4x^2-3)^3 x\,dx.$ Put $u = 4x^2 - 3$, whence $du = 8x\,dx$. The integral then takes the form $\frac{1}{8}\int u^3\,du$. Hence

$$\int (4x^2-3)^3 x\,dx = \tfrac{1}{32}(4x^2-3)^4 + C.$$

EXERCISES

Verify the following:

1. $\int x^4\,dx = \frac{1}{5}x^5 + C.$

2. $\int x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}} + C.$

3. $\int (x^3 - 2x + 5)\,dx = \frac{1}{4}x^4 - x^2 + 5x + C.$

4. $\int (3x^{-2} - 5x^{-4})\,dx = -3x^{-1} + \frac{5}{3}x^{-3} + C.$

Evaluate the following integrals:

5. $\int (ax^{-3} + bx^{-2})\,dx.$

6. $\int (5x^6 - 4x^3 + 2x + 7)\,dx.$

7. $\int (3x^{-2} + 4 - x^2 + 2x^4)\,dx.$

8. $\int \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6}\right)dx.$

9. $\int (x^{\frac{2}{3}} - 3x^{\frac{1}{2}})\,dx.$

10. $\int (2x^2 - 3x)^4\,dx.$

11. $\int (ax+b)^2\,dx.$

12. $\int 3x^2(x^3+4)^2\,dx.$

13. $\int x^3(2x^4-5)^3\,dx.$

14. $\int x(3x^2-7)^4\,dx.$

15. $\int \frac{(3x^2-5)\,dx}{\sqrt{x^3-5x+7}}.$

16. $\int \frac{x\,dx}{(x^2-5)^{\frac{3}{2}}}.$

69. Fundamental integrals. From the results obtained in the preceding chapters we have the following list of fundamental integrals. The student should make himself thoroughly familiar with these formulas.

$$1.\quad \int u^n\,du = \frac{u^{n+1}}{n+1} + C. \quad (n \neq -1).$$

$$2.\quad \int \cos u\,du = \sin u + C.$$

$$3.\quad \int \sin u\,du = -\cos u + C.$$

$$4.\quad \int \sec^2 u\,du = \tan u + C.$$

$$5.\quad \int \csc^2 u\,du = -\cot u + C.$$

$$6.\quad \int \sec u \tan u\,du = \sec u + C.$$

$$7.\quad \int \csc u \cot u\,du = -\csc u + C.$$

$$8.\quad \int \frac{du}{\sqrt{1-u^2}} = \text{arc}\sin u + C = -\text{arc}\cos u + C'.$$

$$9.\quad \int \frac{du}{1+u^2} = \text{arc}\tan u + C = -\text{arc}\cot u + C'.$$

$$10.\quad \int \frac{du}{u\sqrt{u^2-1}} = \text{arc}\sec u + C = -\text{arc}\csc u + C.$$

$$11.\quad \int a^u\,du = \frac{a^u}{\log a} + C.$$

$$12.\quad \int e^u\,du = e^u + C.$$

$$13.\quad \int \frac{du}{u} = \log u + C.$$

The following additional integrals are important and should also be committed to memory.

$$14.\quad \int \frac{du}{u^2-a^2} = \frac{1}{2a}\log\frac{u-a}{u+a} + C. \quad (u^2 > a^2)$$
$$= \frac{1}{2a}\log\frac{a-u}{a+u} + C. \quad (u^2 < a^2).$$

$$15.\quad \int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + C.$$

$$16.\quad \int \tan u\,du = \log \sec u + C.$$

$$17.\quad \int \cot u\,du = \log \sin u + C.$$

$$18.\quad \int \csc u\,du = \log \tan \frac{u}{2} + C.$$

$$19.\quad \int \sec u\,du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + C.$$

The derivation of these last forms will be given later. The student may here verify them by differentiation.

70. Integration by inspection. The integrals tabulated in the preceding article are called **fundamental** or **standard integrals.** In these, the integrands are recognized as the results of previous differentiations; hence, if a given integrand has one of these forms, the integral is known at once as the function previously differentiated. When the integrand has not one of these standard forms, it must be prepared for integration; that is, by suitable transformations, it must be reduced if possible to an integrand of standard form or to a sum of such integrands. The general method used for such reductions forms the subject matter of Chapter XII. In the present chapter we shall consider only such reductions as are quite obvious.

It should be noted that the variable u in the standard forms may be any function, $f(x)$. For example, consider the integral

$$\int e^{\sin x} \cos x \, dx.$$

Since $\cos x \, dx = d(\sin x)$, this may be written

$$\int e^{\sin x} \, d(\sin x),$$

which will be recognized as form 12, with u replaced by $\sin x$.

Frequently the integrand may be made to assume a standard form by the introduction of a constant factor. The following examples are illustrative:

Ex. 1. $$\int (3x-2)^2 dx.$$

Here the standard form $\int u^n du$, where $u = 3x - 2$, and $n = 2$, is suggested. Since $du = d(3x-2) = 3\,dx$, the reduction is effected by introducing the factor 3 in the integrand and the neutralizing factor $\frac{1}{3}$ outside of the integral sign. Thus, we have

$$\int (3x-2)^2 dx = \tfrac{1}{3}\int (3x-2)^2 3\,dx = \tfrac{1}{3}\int (3x-2)^2 d(3x-2) = \tfrac{1}{9}(3x-2)^3 + C.$$

Ex. 2. $$\int e^{ax+b} dx = \frac{1}{a}\int e^{ax+b} a\,dx = \frac{1}{a}\int e^{ax+b} d(ax+b) = \frac{e^{ax+b}}{a} + C.$$

Here the introduction of the factor a reduces the integrand to form 12, u being replaced by $ax + b$.

Ex. 3. $$\int \frac{x\,dx}{\sqrt{a - bx^2}} = \int (a - bx^2)^{-\frac{1}{2}} x\,dx.$$

Introducing the factor $-2b$, we get

$$-\frac{1}{2b}\int (a - bx^2)^{-\frac{1}{2}}(-2\,bx\,dx) = -\frac{1}{2b}\int (a - bx^2)^{-\frac{1}{2}}\,d(a - bx^2).$$

This integral will be recognized as the standard form 1 with $u = a - bx^2$, and $n = -\frac{1}{2}$. The result is therefore

$$-\frac{1}{2b}\cdot\frac{(a - bx^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = -\frac{1}{b}(a - bx^2)^{\frac{1}{2}} + C.$$

Ex. 4. $$\int \frac{dx}{a^2 + b^2x^2}.$$

Looking among the standard forms, it appears that form 9 is the one to which the integrand may be reduced. To get 1 as the first term of the denominator, we divide the denominator by a^2, and thus obtain

$$\frac{1}{a^2}\int \frac{dx}{1 + \frac{b^2}{a^2}x^2} = \frac{1}{a^2}\int \frac{dx}{1 + \left(\frac{b}{a}x\right)^2}.$$

We observe that $\frac{b}{a}x$ is the variable, whence the numerator must take the form $d\left(\frac{b}{a}x\right) = \frac{b}{a}\,dx$. Therefore introducing the factor $\frac{b}{a}$ in the numerator and the neutralizing factor $\frac{a}{b}$ outside the integral sign, we have

$$\frac{a}{b}\cdot\frac{1}{a^2}\int \frac{\frac{b}{a}\,dx}{1 + \left(\frac{b}{a}x\right)^2} = \frac{1}{ab}\arctan\frac{bx}{a} + C.$$

Ex. 5. $$\int \frac{(x - 1)\,dx}{3\,x^2 - 6\,x + 5}.$$

It will be observed that $d(3\,x^2 - 6\,x + 5) = 6(x - 1)\,dx$; hence, except for the factor 6, the numerator of the integrand is the differential of the denominator. By the introduction of this factor the integral reduces to form 13, namely, $\int \frac{du}{u}$. Thus, we have

$$\int \frac{(x-1)dx}{3\,x^2-6\,x+5} = \frac{1}{6}\int \frac{6(x - 1)dx}{3\,x^2-6\,x+5} = \frac{1}{6}\int \frac{d(3\,x^2-6\,x+5)}{3\,x^2-6\,x+5} = \frac{1}{6}\log\,(3\,x^2-6\,x+5) + C.$$

In some cases the integral can be reduced to the algebraic sum of several standard integrals.

Ex. 6. $\int \frac{\sqrt{1+x}}{\sqrt{1-x}}dx = \int \frac{1+x}{\sqrt{1-x^2}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}.$

Now $$\int \frac{dx}{\sqrt{1-x^2}} = \text{arc}\sin x, \qquad \text{(Form 8)}$$

and $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x\,dx) = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}d(1-x^2)$$
$$= -(1-x^2)^{\frac{1}{2}}, \qquad \text{(Form 1)}$$

Hence, $$\int \frac{\sqrt{1+x}}{\sqrt{1-x}}dx = \text{arc}\sin x - \sqrt{1-x^2} + C.$$

If the integrand is a rational fraction having the denominator of the same degree as the numerator, or of lower degree, perform the division before integrating.

Ex. 7. $$\int \frac{x^2-7}{x-3}dx.$$

We have $$\frac{x^2-7}{x-3} = x + 3 + \frac{2}{x-3}.$$

Hence, $$\int \frac{x^2-7}{x-3}dx = \int x\,dx + 3\int dx + 2\int \frac{dx}{x-3}$$
$$= \frac{1}{2}x^2 + 3x + 2\log(x-3) + C.$$

The following examples show that it is often possible to reduce the given integrand to a standard form by writing the denominator as the sum or difference of two squares.

Ex. 8. $\int \frac{dx}{\sqrt{7+6x-x^2}} = \int \frac{d(x-3)}{\sqrt{4^2-(x-3)^2}} = \text{arc}\sin\frac{x-3}{4} + C.$

Ex. 9. $\int \frac{dx}{5-4x-x^2} = \int \frac{d(x+2)}{3^2-(x+2)^2}$
$$= \frac{1}{6}\log\frac{5+x}{1-x} + C, \quad \text{if } (x+2)^2 < 3^2,$$
$$= \frac{1}{6}\log\frac{x+5}{x-1} + C, \quad \text{if } (x+2)^2 > 3^2.$$

Ex. 10. $\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{d(x+2)}{\sqrt{(x+2)^2+4}}$
$$= \log(x+2+\sqrt{x^2+4x+8}) + C.$$

EXERCISES

In the following examples determine by inspection the proper standard form, find the function that replaces u in that form, and perform the integration.

1. $\int (x+a)^4\,dx.$ **2.** $\int 2x(x^2-a^2)^{\frac{1}{2}}\,dx.$ **3.** $\int \frac{6x\,dx}{3x^2+a^2}.$

4. $\int 2xe^{x^2}\,dx.$ **5.** $\int \frac{\cos\theta\,d\theta}{\sin^2\theta}.$ **6.** $\int \cos^2\theta\sin\theta\,d\theta.$

7. $\int \frac{\arctan x\,dx}{1+x^2}.$ **8.** $\int \frac{m\,dx}{\sqrt{1-m^2x^2}}.$ **9.** $\int \sec^2\theta\tan\theta\,d\theta.$

10. Explain why the integration in Ex. 9 may lead to either $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2}\tan^2\theta$.

In the following, reduce the integrands to standard forms by the introduction of proper factors, and integrate.

11. $\int (ax+b^2)^{\frac{5}{2}}\,dx.$ **12.** $\int \frac{x^2\,dx}{2x^3-5}.$ **13.** $\int \frac{x\,dx}{\sqrt{a^2+x^2}}.$

14. $\int \frac{dx}{\sqrt{1-c^2x^2}}.$ **15.** $\int \frac{dx}{x\sqrt{a^2x^2-1}}.$ **16.** $\int a^{m^2x^2}x\,dx.$

In the following, reduce the integrands to standard forms by writing the denominator as the sum or the difference of squares, and integrate.

17. $\int \frac{dx}{x^2+2x+5}.$ **18.** $\int \frac{dx}{\sqrt{9+8x-x^2}}.$

19. $\int \frac{dx}{\sqrt{x^2+6x+10}}.$ **20.** $\int \frac{dx}{11+10x-x^2}.$

Verify the following integrations.

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C = -\arccos\frac{x}{a} + C'.$

22. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C.$

23. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arc\,sec}\frac{x}{a} + C = -\frac{1}{a}\operatorname{arc\,csc}\frac{x}{a} + C'.$

The integrals of exercises 21–23 may be regarded as standard forms and should be memorized.

71. Integration by substitution. The reduction of a given integrand to a standard form is sometimes most easily effected by the

substitution of a new variable. The following examples illustrate this method.

Ex. 1.
$$\int \frac{dx}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}.$$

Let $e^{\frac{x}{2}} = z$; then $dx = 2\frac{dz}{z}$, and the integral takes the form

$$\int \frac{2\,dz}{z(z + z^{-1})} = 2\int \frac{dz}{z^2+1} = 2 \arctan z + C = 2 \arctan e^{\frac{x}{2}} + C.$$

Ex. 2.
$$\int \frac{dx}{x^2\sqrt{1-x^2}}.$$

Let $x = \cos\theta$; then $dx = -\sin\theta\,d\theta$, $\sqrt{1-x^2} = \sin\theta$, and the integral takes the form

$$-\int \frac{\sin\theta\,d\theta}{\cos^2\theta \sin\theta} = -\int \sec^2\theta\,d\theta = -\tan\theta + C = -\frac{\sqrt{1-x^2}}{x} + C.$$

The substitution $x = \frac{1}{z}$ may also be used. The student may work out the details.

As additional exercises, the derivations of forms 14 to 19 are given.

Ex. 3. Since $\dfrac{1}{u^2-a^2} = \dfrac{1}{2a}\left(\dfrac{1}{u-a} - \dfrac{1}{u+a}\right)$, we have, if $u^2 > a^2$,

$$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\int \frac{du}{u-a} - \frac{1}{2a}\int \frac{du}{u+a}$$
$$= \frac{1}{2a}[\log(u-a) - \log(u+a)] = \frac{1}{2a}\log\frac{u-a}{u+a}.$$

The student may derive the second integral of form 14 and show that it applies when $u^2 < a^2$.

Ex. 4. To integrate $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}}$, put $u^2 \pm a^2 = z^2$. Then $2\,u\,du = 2\,z\,dz$,

whence
$$\frac{du}{z} = \frac{dz}{u} = \frac{du+dz}{u+z}.$$

Therefore,
$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \int \frac{du}{z} = \int \frac{du+dz}{u+z} = \log(u+z)$$
$$= \log(u + \sqrt{u^2 \pm a^2}).$$

Ex. 5.
$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d(\cos x)}{\cos x}$$
$$= -\log\cos x + C = \log\sec x + C.$$

Ex. 6.
$$\int \cot x\, dx = \int \frac{\cos x\, dx}{\sin x} = \int \frac{d(\sin x)}{\sin x} = \log \sin x + C.$$

Ex. 7.
$$\int \csc x\, dx.$$

Let $z = \tan \frac{x}{2}$; then $\csc x = \frac{1+z^2}{2z}$, and $dx = \frac{2\,dz}{1+z^2}$.

Substituting these values, we get

$$\int \csc x\, dx = \int \frac{dz}{z} = \log z + C = \log \tan \frac{x}{2} + C.$$

Ex. 8.
$$\int \sec x\, dx.$$

Making the same substitution as in Ex. 7, the integral reduces to the form $\int \frac{2\,dz}{1-z^2}$. Hence, we have

$$\int \sec x\, dx = 2\int \frac{dz}{1-z^2} = \log \frac{1+z}{1-z} + C = \log \frac{1+\tan \frac{x}{2}}{1-\tan \frac{x}{2}} + C$$
$$= \log \tan \left[\frac{x}{2} + \frac{\pi}{4}\right] + C.$$

EXERCISES

1. $\int \frac{x\,dx}{a+bx}$, let $z = a + bx$.

2. $\int \frac{dx}{x^2\sqrt{x^2-a^2}}$, let $x = a \sec \theta$.

3. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$, let $x = a \sin \theta$.

4. $\int \frac{x\,dx}{\sqrt{x-2}}$.

5. $\int \frac{dx}{x\sqrt{2x+1}}$, let $z^2 = 2x + 1$.

6. Verify the integrations given in examples 21, 22, and 23 of the preceding article by the substitutions $x = a \sin \theta$, $x = a \tan \theta$, and $x = a \sec \theta$, respectively.

7. Evaluate the integral $\int \frac{dx}{\sqrt{2ax - x^2}}$ by means of the substitution $x = a(1 - \cos \theta)$.

72. Character of the integration process. There is a fundamental difference between differentiation and integration. The former is a direct, the latter an inverse process. As we have seen, when once the derivatives of the elementary functions have been found, the derivatives of any functions expressed in terms

of these functions can usually be obtained by one or more direct operations. On the contrary, it is not, in general, possible to determine the integral of a function from the known integrals of the elementary functions in terms of which the given function is expressed. There are no general methods for expressing the integral of a product or a quotient of functions, or of a function of a function in terms of the integrals of the component functions. Hence the process of integration consists, not in a series of direct operations carried out in accordance with a general method, but rather in attempts to reduce the given function to a form, the integral of which is known.

Integration, then, is largely a question of judgment in attacking the problem. Specific rules cannot be given, but the following general directions will be found useful. First inspect the integrand carefully and determine whether it is a standard form. Frequently the integrand is merely a disguised standard form and needs only a rearrangement of its terms to be recognized as such. If the integrand is not in a standard form, see if it cannot be made to assume such a form by the introduction of a constant factor. Do not neglect the neutralizing factor outside the integral sign. If this device does not appear to be effective, try the substitution of a new variable.

MISCELLANEOUS EXERCISES

Integrate the following:

1. $\int mx^{a-3}\,dx.$

2. $\int (3x - x^2)^{-\frac{1}{2}}(3 - 2x)\,dx.$

3. $\int \frac{x^2\,dx}{(x^3 - a^3)^{\frac{3}{2}}}.$

4. $\int \frac{(x-3)\,dx}{x^2 - 6x + 1}.$

5. $\int \frac{(x-3)\,dx}{\sqrt{x^2 - 6x + 1}}.$

6. $\int xe^{x^2}\,dx.$

7. $\int \cos^2\theta \sin\theta\,d\theta.$

8. $\int \frac{\cos\theta\,d\theta}{\sin^3\theta}.$

9. $\int \frac{(2 + 3x)\,dx}{\sqrt{4 - x^2}}.$

10. $\int \frac{dx}{(ax + b)^{\frac{3}{2}}}.$

11. $\int \frac{x^2\,dx}{3x^3 - 5}.$

12. $\int \frac{2x^3 - 5x + 1}{x^4 - 5x^2 + 2x - 7}\,dx.$

13. $\int \frac{dx}{x^2 + 6x + 13}$.

14. $\int \frac{x^3\,dx}{x + 1}$.

15. $\int \frac{2x^2 + 1}{x^3 + x}\,dx$.

16. $\int \frac{\sin x\,dx}{a + b\cos x}$.

17. $\int e^{\cos x}\sin x\,dx$.

18. $\int \frac{1 - x}{1 + x^2}\,dx$.

19. $\int \frac{dx}{1 + 3x + 2x^2}$.

20. $\int \frac{dx}{\sqrt{12 - 3x - x^2}}$.

21. $\int \frac{dx}{\sqrt{x^2 + 6x + 1}}$.

22. $\int \frac{dz}{z^2 - 6}$.

23. $\int \frac{d\theta}{4 - 3\theta^2}$.

24. $\int \frac{dx}{x\sqrt{1 - \log^2 x}}$.

25. $\int \frac{ds}{\sqrt{2as + s^2}}$.

26. $\int \frac{du}{\sqrt{(u + m)^2 - n^2}}$.

27. $\int \frac{x^3\,dx}{\sqrt{a^8 - x^8}}$.

28. $\int \frac{\sqrt{1 - x}}{\sqrt{1 + x}}\,dx$.

29. $\int \frac{e^{-x}\,dx}{1 + e^{-x}}$.

30. $\int \frac{(x + 2)\,dx}{x\sqrt{x}}$.

31. Given $y = \int \frac{x\,dx}{\sqrt{a^2 - x^2}}$; knowing that $y = 0$ when $x = 0$, find the integral, determine the constant of integration, and thus find the relation between x and y.

32. In the preceding example, find the relation between x and y if $y = 0$ when $x = a$.

33. A curve passes through the point (0, 2), and the general expression for its slope is $\frac{x}{x^2 + 1}$. Find the equation of the curve.

34. Find the equation of a curve that passes through the point (1, 0) and whose slope is given by the expression $\frac{a}{x^2} + bx$.

35. In a certain type of motion the relation $\frac{ds}{\sqrt{v_0^2 - k^2s^2}} = dt$ holds, v_0 being the initial velocity. Express the distance s as a function of the time t, knowing that $s = 0$ when $t = 0$.

CHAPTER VII

SIMPLE APPLICATIONS OF INTEGRATION

73. Curves having given properties. It has been shown that the value of the derivative of a function $y = f(x)$ for a particular value of the variable gives the slope of the tangent to the curve representing the function for the value of the variable. Furthermore, it has been shown that the derivative enters into the equations of the tangent and normal to the curve and into the expressions for the length of the tangent, the length of the normal, the subtangent, and the subnormal. We have so far considered problems in which from a given function the derivative was found, and from this derivative the desired property of the curve was obtained. We shall now consider inverse problems in which a property of the curve is given, and from it the equation of the curve is to be obtained. These problems naturally involve the evaluation of an integral.

Ex. 1. Determine the equation of a curve at every point of which the slope is equal numerically to one half the abscissa.

We have here
$$D_x y = \tfrac{1}{2}x,$$
whence
$$y = \int \tfrac{1}{2}x\,dx = \tfrac{1}{4}x^2 + C.$$

This is the equation of a parabola having the Y-axis as the axis of the curve. By giving the constant C different values, we get a series of parabolas having the same slope for the same value of x.

Ex. 2. Find the equation of the curve whose polar subtangent has a constant length a.

In this case, we have
$$\rho^2 D_\rho \theta = a,$$
or
$$D_\rho \theta = \frac{a}{\rho^2}.$$
Therefore,
$$\theta = \int a \frac{d\rho}{\rho^2} = -\frac{a}{\rho} + C,$$
or
$$\rho = \frac{a}{C - \theta}.$$

EXERCISES

1. Find the equations of the curves whose slopes are respectively:

$$3;\ \tfrac{1}{2}x^2;\ mx;\ -\frac{1}{x^2};\ ax-b;\ \frac{1}{ax^3}.$$

2. Find the equation of the curve whose slope at any point is double the abscissa at that point.

3. Find the equation of a curve whose subnormal is b times the ordinate.

4. Find the equation of the curve whose subnormal varies inversely as the ordinate.

5. Find the polar equation of the curve for which $\tan\psi = k\rho$.

6. Find the polar equation of the curve whose polar subnormal has a constant length m.

7. Find the equation of the curve which passes through the point (2, 3) and whose tangent has the slope $3x+5$.

8. Find the equation of the curve whose slope at any point is proportional to the ordinate at that point.

9. Find the equation of the curve whose subtangent has the constant value a.

10. Find the polar equation of a curve for which the angle ψ is constant.

74. Rectilinear motion. The relations between the time, speed, acceleration, and space traversed by a point moving in a straight line are:

$$v = D_t s, \quad a = D_t v.$$

We have, therefore, the inverse relations

$$v = D_t^{-1}a = \int a\,dt, \tag{1}$$

$$s = D_t^{-1}v = \int v\,dt. \tag{2}$$

In the direct problem, starting with the space traversed as a function of the time, we were able, by taking derivatives, to find the speed and acceleration. In the inverse problem, having the acceleration given as a function of the time, we can by integration determine first the speed, then the space traversed.

A relation that is useful in certain problems is the following:

Since
$$a = \frac{dv}{dt} \quad\text{and}\quad v = \frac{ds}{dt},$$

we get by eliminating dt,

$$v\,dv = a\,ds, \tag{3}$$

whence $$v^2 = 2\int a\,ds. \tag{4}$$

Ex. 1. A point moves in a straight line with a constant acceleration a. The speed and the space passed over at the end of a given time-interval are required.

We have $$v = \int a\,dt = at + C. \tag{a}$$

To determine the constant C, let v_0 denote the initial speed, that is, the speed when $t = 0$. Substituting these values in (a), we have

$$v_0 = 0 + C, \text{ or } C = v_0;$$

hence, (a) becomes $$v = at + v_0. \tag{b}$$

We have further $$s = \int v\,dt = \int (at + v_0)\,dt,$$

whence, performing the integration, we obtain

$$s = \tfrac{1}{2}at^2 + v_0 t + C'. \tag{c}$$

To determine the constant C', let s_0 denote the initial space, that is, the space when $t = 0$. Substituting in (c), we have

$$s_0 = 0 + C', \text{ or } C' = s_0,$$

and we have finally $$s = \tfrac{1}{2}at^2 + v_0 t + s_0. \tag{d}$$

An important application of these laws is to the case of freely falling bodies. By observation, it is found that a body falling freely in a vacuum at a given point on the earth's surface has a constant acceleration. This acceleration, while constant for any one place, varies for different localties within small limits. Its value may be taken as 32.2 ft./sec.^{2}.

If the body falls from rest, we have $v_0 = 0$ and $s_0 = 0$. Denoting by g the constant acceleration, we have, therefore,

$$v = gt, \tag{5}$$

$$s = \tfrac{1}{2}gt^2. \tag{6}$$

Eliminating t between (5) and (6), we get as a third relation,

$$v^2 = 2\,gs,$$

or $$v = \sqrt{2\,gs}, \tag{7}$$

which might also be obtained from (4).

Ex. 2. Investigate the motion of a body projected vertically upward from the earth's surface with an initial velocity of b feet per second.

We have here $v_0 = b$ and $s_0 = 0$. The acceleration is g, but in this case is opposite to the direction of motion; hence $a = -g$. From the fundamental relations (1) and (2), we have upon integration

$$v = -gt + b,$$

$$s = -\tfrac{1}{2}gt^2 + bt.$$

The body will reach its highest point when $v = 0$, that is, when

$$0 = -gt + b, \text{ or } t = \frac{b}{g},$$

and the height h to which it rises is found by substituting this value of t in the second equation. Thus

$$h = -\tfrac{1}{2}g\left(\frac{b}{g}\right)^2 + b\cdot\frac{b}{g} = \frac{b^2}{2g}.$$

75. Rotation about a fixed axis. Corresponding to the direct relations

$$\omega = D_t\theta, \qquad \alpha = D_t\omega,$$

we have the inverse relations

$$\omega = D_t^{-1}\alpha = \int \alpha\, dt, \tag{1}$$

$$\theta = D_t^{-1}\omega = \int \omega\, dt, \tag{2}$$

and starting with the angular acceleration α as a given function of the time, we can by integration determine the angular speed and the angle swept through by a given radius. For the case in which the angular acceleration is a constant, say α_0, we readily derive the following formulas for ω and θ:

$$\omega = \alpha_0 t + \omega_0. \tag{3}$$

$$\theta = \tfrac{1}{2}\alpha_0 t^2 + \omega_0 t + \theta_0. \tag{4}$$

As in Art. 74, we have also the useful relation

$$\omega\, d\omega = \alpha\, d\theta. \tag{5}$$

76. Motion of a projectile. A body is thrown with an initial velocity v_0 at an angle α with the horizontal. Neglecting the resistance of the atmosphere, the path of the body is to be determined.

Let the plane containing the path be taken as the XY-plane, with the Y-axis vertical. Then the X- and Y-components of the initial velocity v_0 are respectively $v_0 \cos \alpha$ and $v_0 \sin \alpha$. The

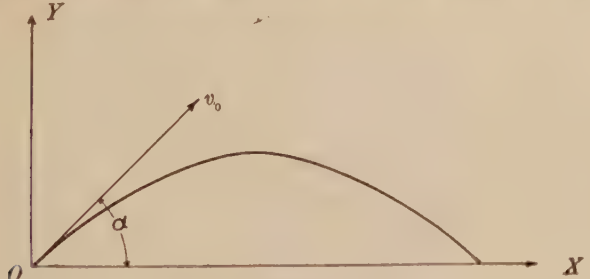

Fig. 25.

X-component remains constant throughout the motion, that is, the X-component a_x of the acceleration of the body is zero. The Y-component a_y of the acceleration is $-g$, as in the case of a body projected vertically upward. We have therefore

$$a_x = \frac{dv_x}{dt} = 0, \qquad v_x = C_1 = v_0 \cos \alpha. \tag{1}$$

$$a_y = \frac{dv_y}{dt} = -g, \quad v_y = -gt + C_2. \tag{2}$$

To determine C_2, we have at the beginning of the motion $v_y = v_0 \sin \alpha$, when $t = 0$; hence $C_2 = v_0 \sin \alpha$, and $v_y = v_0 \sin \alpha - gt$. To find the X- and Y-coördinates of the position of the body at the time t, we have

$$v_x = \frac{dx}{dt} = v_0 \cos \alpha,$$

whence $$x = v_0 t \cos \alpha \tag{3}$$
where the constant of integration is zero.

Similarly, $$v_y = \frac{dy}{dt} = v_0 \sin \alpha - gt,$$

whence $$y = v_0 t \sin \alpha - \tfrac{1}{2} g t^2. \tag{4}$$

Eliminating t between (3) and (4), the resulting equation,

$$y = x \tan \alpha - \frac{gx^2}{2\, v_0^2 \cos^2 \alpha}, \tag{5}$$

is the equation of the path.

EXERCISES

1. Find an expression for the velocity v and distance s when the acceleration is given by the relation:

(a) $a = m - nt^2$. (b) $a = -mk^2 \cos kt$. (c) $a = \frac{kv_0}{2}(e^{kt} - e^{-kt})$.

2. If the speed of a point moving in a straight line is given by the relation $v = 10t - t^2$, find expressions for the acceleration and the distance traversed, starting from rest, at the end of t_1 seconds.

3. In Ex. 2 find the time that will elapse before the point comes to rest, the distance the point has moved in that time, and the final acceleration.

4. A hoisting drum has an angular speed $\omega_0 = 24$ rad./sec. A brake is applied in such a way as to produce a retardation (negative acceleration) of 3 rad./sec. Find the time required to bring the drum to rest and the number of revolutions made after the brake is applied.

5. Show that the *range* of a projectile on a horizontal plane is

$$R = \frac{v_0^2}{g} \sin 2\alpha.$$

6. Show that the velocity of a projectile at any instant is $v = \sqrt{v_0^2 - 2gy}$.

7. A stone is thrown horizontally with an initial velocity v_0 from the top of a cliff. Find the path followed.

8. If the top of the cliff is 100 feet above the level of a stream 150 feet wide at its base, what initial velocity is required to carry the stone across the stream?

77. Harmonic motion. According to the laws of mechanics, the acceleration of a body of given mass is proportional to the force acting on it. Hence, since it is usually the force that is known, the character of the motion is specified by the law according to which the acceleration changes; and the velocity and distance traversed are obtained by one and two integrations, respectively. The acceleration a may be given as a function of t, of v, or of s. In this article and those immediately following we shall consider a few important examples of motion following various laws.

When a point moves in such a way that the negative acceleration is proportional to the distance of the point from a fixed origin O, the motion is said to be **simple harmonic**. In this case, we have

$$a = -k^2 s, \tag{1}$$

whence from the relation $v\,dv = a\,ds$, [Art. 74, Eq. (3)]

$$v\,dv = -k^2 s\,ds.$$

Integrating, we obtain

$$v^2 = C - k^2 s^2.$$

To determine the constant C, let v_0 denote the velocity at the origin O; then $v = v_0$ when $s = 0$, and consequently $C = v_0^2$. Hence

$$v^2 = v_0^2 - k^2 s^2. \qquad (2)$$

Evidently $v = 0$ when $v_0^2 = k^2 s^2$, that is, when $s = \pm \frac{v_0}{k}$. It follows that the point oscillates between the points $\frac{v_0}{k}$ and $-\frac{v_0}{k}$ equidistant from the origin.

The relation between the distance traversed and the time is obtained from the relation

$$v = \frac{ds}{dt} = \sqrt{v_0^2 - k^2 s^2},$$

whence

$$\frac{ds}{\sqrt{v_0^2 - k^2 s^2}} = dt.$$

Integrating, we have

$$\frac{1}{k} \operatorname{arc\,sin} \frac{ks}{v_0} = t + C. \qquad (3)$$

If we make $t = 0$ when the point is at the origin and $s = 0$, then $C = 0$, and we have

$$s = \frac{v_0}{k} \sin kt. \qquad (4)$$

From (4), as from (2), we can see the vibratory character of the motion; for as kt takes successively the values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, etc., $\sin kt$ takes the values $0, 1, 0, -1, 0$, etc., and s the values $0, \frac{v_0}{k}, 0, -\frac{v_0}{k}, 0$, etc.

78. Motion in a resisting medium. When a body moves in a medium as air or water, it encounters a resistance that is dependent upon the velocity v. For small values of v, this resistance is approximately proportional to the first power of v, while for higher velocities it is more nearly proportional to v^2.

1. Let a body having an initial velocity v_0 be subjected to a resistance proportional to v; then the acceleration is $-kv$, where k is a constant, and we have

$$a = \frac{dv}{dt} = -kv, \tag{1}$$

whence
$$dt = -\frac{dv}{kv},$$

and
$$t = \frac{1}{k}\log\frac{1}{v} + C_1. \tag{2}$$

To determine C_1, we have $v = v_0$ when $t = 0$. Therefore $C_1 = \frac{1}{k}\log v_0$, and (2) becomes

$$t = \frac{1}{k}\log\frac{v_0}{v}, \tag{3}$$

whence
$$v = v_0 e^{-kt}. \tag{4}$$

To find a relation between s and t, we have from (4)

$$\frac{ds}{dt} = v_0 e^{-kt},$$

whence
$$s = -\frac{v_0}{k}e^{-kt} + C_2. \tag{5}$$

Putting $s = 0$, when $t = 0$, we find $C_2 = \frac{v_0}{k}$, whence (5) becomes

$$s = \frac{v_0}{k}(1 - e^{-kt}).$$

2. A body falling from a height to the earth encounters a resistance from the atmosphere proportional approximately to the square of the velocity. The acceleration without resistance being g, the acceleration when the resistance is taken into account is evidently $g - k^2v^2$. We have then

$$\frac{dv}{dt} = g - k^2v^2, \tag{6}$$

also from the relation $v\,dv = a\,ds$,

$$v\,dv = (g - k^2v^2)\,ds. \tag{7}$$

From (6), $$t = \int\frac{dv}{g - k^2v^2} = \frac{1}{2k\sqrt{g}}\log\frac{\sqrt{g} + kv}{\sqrt{g} - kv} + C_1, \tag{8}$$

and from (7) $s = \int \frac{v\,dv}{g - k^2v^2} = -\frac{1}{2k^2}\log(g - k^2v^2) + C_2.$ (9)

If we take the initial velocity as 0 when $s = 0$ and $t = 0$, we find the constants to be $C_1 = 0$, $C_2 = \frac{1}{2k^2}\log g$. Hence from (8), we get

$$v = \frac{\sqrt{g}}{k}\left(\frac{e^{2kt\sqrt{g}} - 1}{e^{2kt\sqrt{g}} + 1}\right), \tag{10}$$

and from (9)

$$s = \frac{1}{2k^2}\log\frac{g}{g - k^2v^2}, \tag{11}$$

whence

$$v^2 = \frac{g}{k^2}(1 - e^{-2k^2s}). \tag{12}$$

As s increases indefinitely, the velocity v approaches the limiting value $\frac{\sqrt{g}}{k}$. Thus a body falling from a great height, as a raindrop, approaches the earth with practically this limiting velocity. Observe that the same result is obtained from (10) when t is increased indefinitely.

EXERCISES

1. Show that the time of a complete oscillation in harmonic motion is $\frac{2\pi}{k}$.

2. If a point moves in a circle with constant speed, the projection of the point on a diameter moves on the diameter according to the law of simple harmonic motion. Prove this statement, and show that the constant k in the equation of Art. 77 is the angular speed of the radius that joins the moving point to the center.

3. A man is rowing in still water at a speed of m ft. per second. If he suddenly stops rowing, find the law according to which the boat continues to move, assuming that the resistance of the water is proportional to the speed.

4. In Ex. 3 find the time that must elapse before the boat comes to rest. Give a reason for the absurdity of the result.

5. Show that the speed in harmonic motion is expressed as a function of the time by the equation $v = v_0 \cos kt$.

6. If the retarding effect of fluid friction on a rotating disk is proportional to the angular speed ω, that is, if $\frac{d\omega}{dt} = -k\omega$, show that $\omega = \omega_0 e^{-kt}$, where ω_0 is the initial angular speed.

7. Find an expression for ω when the retardation is proportional to the square of the angular speed ω, which is approximately true for a very rapidly revolving disk, as a gyroscope.

79. Physical problems involving exponential functions. If a function has the general form

$$y = e^{ax},$$

then
$$D_x y = ae^{ax} = ay;$$

that is, the rate of change of the function is proportional to the function itself. Many natural phenomena follow this law, which has been called by Lord Kelvin the **compound interest law**. One example of this law has been shown in the motion of a body in a resisting medium. The following are other illustrations:

(a) *Atmospheric pressure.* It is well known that the pressure of the atmosphere decreases as the distance from the earth's surface increases. Assuming the temperature of the atmosphere to be constant, the rate of change of pressure with the height at any given height is proportional to the pressure at that height; that is,

$$\frac{dp}{dh} = -kp.$$

The negative sign is used because p decreases as h increases. We have then

$$\frac{dp}{p} = -k\,dh,$$

and integrating, the result is

$$\log p = -kh + C.$$

To determine the constant of integration C, let p_0 denote the pressure at the earth's surface where $h = 0$. Then

$$\log p_0 = 0 + C, \text{ or } C = \log p_0.$$

Substituting this value of C and transposing, we have

$$kh = \log p_0 - \log p = \log \frac{p_0}{p},$$

whence
$$\frac{p_0}{p} = e^{kh},$$

or
$$p = p_0 e^{-kh}.$$

This formula gives the pressure at any definite height, when the constant k is known.

(b) *Newton's law of cooling.* A body has a temperature τ which is higher than the temperature τ_0 of the surrounding medium. The rate at which it cools is approximately proportional to the difference in temperature $\tau - \tau_0$; that is, approximately,

$$\frac{d\tau}{dt} = -k(\tau - \tau_0),$$

where k is some constant. We have then

$$\frac{d\tau}{\tau - \tau_0} = -k\,dt,$$

whence, by integration,

$$\log(\tau - \tau_0) = -kt + C.$$

Let τ_1 denote the temperature when $t = 0$; then

$$C = \log(\tau_1 - \tau_0),$$

and substituting this value of C, we get

$$\log\frac{\tau_1 - \tau_0}{\tau - \tau_0} = kt,$$

or

$$\frac{\tau_1 - \tau_0}{\tau - \tau_0} = e^{kt}.$$

Hence, we have

$$\tau = \tau_0 + (\tau_1 - \tau_0)e^{-kt},$$

from which τ can be found for any time interval.

(c) *Inversion of sugar.* Cane sugar in solution is decomposed into other substances through the presence of acids. The rate at which the process takes place is proportional to the mass of sugar still unchanged. Thus, if s is the original mass of sugar and x is the mass inverted, the rate of inversion at this stage of the process is proportional to the unchanged mass $s - x$. The equation expressing this law is

$$\frac{dx}{dt} = k(s - x).$$

Integrating, we get

$$-\log(s - x) = kt + C.$$

To determine the constant of integration C, we make use of the fact that $x = 0$ when $t = 0$, whence $C = -\log s$. Substituting this value of C in the original equation, we have

$$\log\frac{s}{s - x} = kt,$$

or

$$\frac{s}{s - x} = e^{kt}.$$

Solving for x, we get

$$x = s(1 - e^{-kt}).$$

MISCELLANEOUS EXERCISES

Integrate the following.

1. $\int \frac{x\,dx}{x^4+3}$. 2. $\int \frac{x^4\,dx}{x^2-1}$. 3. $\int \frac{e^{2x}\,dx}{\sqrt{e^x+1}}$.

4. $\int \frac{dx}{e^x+e^{-x}}$. 5. $\int \frac{3\,dx}{x\sqrt{4x^2-7}}$. 6. $\int \frac{dx}{(x-a)\sqrt{(x-a)^2-b^2}}$.

7. Determine curves whose slopes are respectively

$$(a)\ 3x+2;\quad (b)\ x^{\frac{1}{3}}+c;\quad (c)\ \cos mx;\quad (d)\ e^{cx}.$$

Assume a point on each curve and thus determine the constant of integration in each case.

8. Find the equation of a curve whose tangent has the slope $4-x^2$ and which passes through the point $(3, -2)$.

9. Since the specific heat c is given by the derivative $\frac{dQ}{d\tau}$, it follows that $Q=\int c\,d\tau$. (See Art. 19.) From the experimental law, $c=a+b\tau+c\tau^2$, derive an expression for Q.

10. The equation $E=Ri+L\frac{di}{dt}$ expresses for a constant electromotive force E the relation between the time t and the current i in a circuit in which an electric current is flowing. R denotes the resistance and L the inductance of the circuit, and both are constants. If $i=0$ when $t=0$, show that the value of i at the time t_1 is $\frac{E}{R}(1-e^{-\frac{Rt_1}{L}})$.

11. If, in Ex. 10, i_0 is the value of the current at a given instant, and the electromotive force E is removed at this instant, show that at the end of t_1 seconds after the removal the value of the current is $i_1=i_0e^{-\frac{Rt_1}{L}}$.

SUGGESTION. Make $E=0$, and take $t=0$ when $i=i_0$.

12. If the acceleration of a point moving in a straight line is given by the relation $a=k^2s$, show that the relation between the distance and the time is $s=\frac{v_0}{2k}(e^{kt}-e^{-kt})$, where v_0 denotes the initial speed.

13. A curve is described by a point moving along a radius vector while the radius is rotating about a pole. If the velocity along the radius is n times the velocity at right angles to it, what is the polar equation of the curve?

14. In the theory of the bending of beams, x-anti-derivatives of the following functions are required. Find these anti-derivatives and determine the constants of integration from the conditions given.

(a) $f(x) = \frac{1}{2}w(L-x)^2$, $\quad D^{-1}f(x) = 0$, when $x = 0$.

(b) $f(x) = L^2x - Lx^2 + \frac{1}{3}x^3$, $\quad D^{-1}f(x) = 0$, when $x = 0$.

(c) $f(x) = \frac{1}{8}wL^2x - \frac{1}{6}wx^3$, $\quad D^{-1}f(x) = 0$, when $x = 0$.

(d) $f(x) = \frac{3}{8}wLx - \frac{1}{2}wx^2$, $\quad D^{-1}f(x) = 0$, when $x = L$.

CHAPTER VIII

SUCCESSIVE DIFFERENTIATION AND INTEGRATION

80. Definition of the nth derivative. The derivative of a function is, in general, a function of the same independent variable as the original function, and it may itself have a derivative if it fulfills the necessary conditions. The differentiation of the original function produces the first derivative; the differentiation of this first derivative gives the second derivative; the result of differentiating this second derivative is the third derivative; and so on. If the process is repeated n times, the final result is the nth derivative of the function with respect to its independent variable.

As an example, consider the function

$$f(x) = x^3 - 2x + 5.$$

The first derivative is $\quad 3x^2 - 2,$

which is also a function of x. Differentiating this, the second derivative is $6x$, which is again a function of x. The third derivative is 6, a constant, and the fourth and each successive derivative is zero.

If $y = f(x)$ is the original function, the successive derivatives are also denoted by the symbols:

$$D_x y,\ D_x^2 y,\ D_x^3 y,\ \cdots,\ D_x^n y,$$

$$f'(x),\ f''(x),\ f'''(x),\ \cdots,\ f^n(x),$$

or by $\quad \dfrac{dy}{dx},\ \dfrac{d^2y}{dx^2},\ \dfrac{d^3y}{dx^3},\ \cdots,\ \dfrac{d^ny}{dx^n}.$

The symbols $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$, etc., are the ones most frequently used. The form of these symbols may be explained as follows: Since

$\dfrac{d\,f(x)}{dx}=f'(x)$, we may regard $\dfrac{d}{dx}$ like D_x as an operator which, when applied to $f(x)$, produces the derivative $f'(x)$. If

$$y=f(x),$$

we have, therefore,

$$\frac{d}{dx}y=\frac{dy}{dx}=f'(x),$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=f''(x),$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]=f'''(x), \text{ etc.}$$

For the sake of convenience in writing

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ is abbreviated to } \frac{d^2y}{dx^2},\ \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] \text{ to } \frac{d^3y}{dx^3}, \text{ etc.}$$

A general formula for the nth derivative may be found for certain functions as shown by the following examples.

Ex. 1. Let $y=x^p$, where p is a positive integer.

Then

$$D_xy=px^{p-1},$$
$$D_x^2y=p(p-1)x^{p-2},$$
$$D_x^3y=p(p-1)(p-2)x^{p-3},$$
$$\cdots\cdots\cdots\cdots$$
$$D_x^ny=p(p-1)(p-2)\cdots(p-n+1)x^{p-n}.$$

For $n=p$, $x^{p-n}=1$ and $D_x^p(x^p)=p\,!$; hence for $n>p$, the derivative is zero.

Ex. **2.** Let $y=e^{ax}$.

We have

$$D_xy=ae^{ax},$$
$$D_x^2y=a^2e^{ax},$$
$$D_x^3y=a^3e^{ax},$$
$$\cdots\cdots$$
$$D_x^ny=a^ne^{ax}.$$

81. Successive differentiation of implicit functions. The following example illustrates the method of procedure in finding successive derivatives of an implicit function.

Ex. Given $f(x, y) = x^2y + 5y - 3x = 0$; find $\frac{d^2y}{dx^2}$.

Differentiating with respect to x, we obtain

$$2xy + x^2\frac{dy}{dx} + 5\frac{dy}{dx} - 3 = 0, \tag{1}$$

whence

$$\frac{dy}{dx} = \frac{3 - 2xy}{x^2 + 5}. \tag{2}$$

A second differentiation with respect to x gives

$$\frac{d^2y}{dx^2} = \frac{(x^2+5)\frac{d}{dx}(3-2xy) - (3-2xy)\frac{d}{dx}(x^2+5)}{(x^2+5)^2}$$

$$= \frac{-(x^2+5)\left(2y + 2x\frac{dy}{dx}\right) - (3-2xy)2x}{(x^2+5)^2}. \tag{3}$$

Substituting in (3) the expression for $\frac{dy}{dx}$ given in (2), we obtain after reduction

$$\frac{d^2y}{dx^2} = \frac{6x^2y - 12x - 10y}{(x^2+5)^2}. \tag{4}$$

82. Geometrical and physical interpretations of the second derivative. As has been shown, the first derivative D_xy $\left(\text{or } \frac{dy}{dx}\right)$ gives the rate of change of y with respect to x, and expresses geometrically the slope of the curve which is the graph of $y = f(x)$. Evidently, the second derivative D_x^2y $\left(\text{or } \frac{d^2y}{dx^2}\right)$ gives the rate of change of the slope compared with the rate of change of the abscissa x. Take, for example, the function $y = mx + b$, which is represented by a straight line. The slope is constant, and therefore its rate of change is zero. This is shown by the derivatives; for $D_xy = m$ and $D_x^2y = 0$.

In the case of a moving point, the speed v is the derivative D_ts, while the tangential acceleration a is D_tv. If now D_ts be substituted for v, we have

$$a = D_t(D_ts) = D_t^2s,$$

or

$$a = \frac{d^2s}{dt^2}; \tag{1}$$

that is, *the tangential acceleration is the second time-derivative of the space s.*

In the case of rotation about a fixed axis, the angular speed is $D_t\theta = \frac{d\theta}{dt} = \omega$, and the angular acceleration is

$$\alpha = D_t\omega = D_t(D_t\theta) = D_t^2\theta = \frac{d^2\theta}{dt^2}; \tag{2}$$

that is, *the angular acceleration is the second time-derivative of the angle swept over.*

EXERCISES

Find the second derivatives of the following functions.

1. $y = x^3 + 4x - 7.$ 2. $y = x^x.$ 3. $y = e^{ax}\cos x.$

4. $y = \text{arc}\cos x.$ 5. $y = \sqrt{a^2 - x^2}.$ 6. $y = \log(a^2 + x^2).$

Find $\frac{d^3y}{dx^3}$ for the following.

7. $y = xe^x.$ 8. $y = \log(x - 3).$ 9. $y = x \sin x.$

10. If $y = \sin x + \cos x$, show that $y = \frac{d^4y}{dx^4} = \frac{d^8y}{dx^8}.$

Find the nth derivatives of

11. $y = a^x.$ 12. $y = \frac{1}{x}.$ 13. $y = \log x.$

14. Find $\frac{d^2y}{dx^2}$ from (a) the equation of the parabola $y^2 = 4px$; (b) the equation of the ellipse $a^2y^2 + b^2x^2 = a^2b^2.$

Find $\frac{d^2y}{dx^2}$ for the following implicit functions.

15. $x^3y + 3x^2y^2 + xy^3 = 0.$ 16. $y^2 - 2axy + x^2 - c = 0.$

17. $xe^y - c = 0.$

18. Find $\frac{d^2r}{d\theta^2}$ from $r = \frac{1}{1 - \cos\theta}.$

19. Find $\frac{d^3r}{d\theta^3}$ from $r = \log \sin \theta.$

20. If $pv^n = C$, show that $\frac{d^2p}{dv^2} = n(n+1)\frac{p}{v^2}.$

21. If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$

22. If $y = \log[x + \sqrt{a^2 + x^2}]$, show that $\frac{d^2y}{dx^2} + \frac{x}{a^2 + x^2}\frac{dy}{dx} = 0.$

83. Successive integration. As the inverse of successive differentiation, we have the process of successive integration. Starting with a function $y = f(x)$, considered as an nth derived function, a single integration gives a new function, the integral. The integration of this second function gives a second integral, and so on. The result of n integrations is the nth integral of the given function.

For example, let $$f(x) = 5x^2.$$

Then $$\int 5x^2dx = \tfrac{5}{3}x^3 + C_1 \text{ is the first integral,}$$

$$\int (\tfrac{5}{3}x^2 + C_1)dx = \tfrac{5}{12}x^4 + C_1x + C_2 \text{ is the second integral,}$$

$$\int (\tfrac{5}{12}x^4 + C_1x + C_2)dx = \tfrac{1}{12}x^5 + \tfrac{1}{2}C_1x^2 + C_2x + C_3 \text{ is the third integral,}$$

etc.

It will be observed that an integral contains the number of arbitrary constants indicated by its order; thus the third integral has three, the fourth, four, and so on.

Let the successive integrals of $f(x)$ be denoted by $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$, respectively; we then have

$$f_1(x) = \int f(x)\,dx;$$

$$f_2(x) = \int f_1(x)\,dx;$$

$$f_3(x) = \int f_2(x)\,dx, \text{ etc.}$$

Ex. 1. Find the successive integrals of $y = e^{ax}$.

$$f_1(x) = \int f(x)dx = \int e^{ax}dx = \frac{1}{a}e^{ax} + C_1,$$

$$f_2(x) = \int \left(\frac{1}{a}e^{ax} + C_1\right)dx = \frac{1}{a^2}e^{ax} + C_1x + C_2,$$

$$f_3(x) = \int \left(\frac{1}{a^2}e^{ax} + C_1x + C_2\right)dx = \frac{1}{a^3}e^{ax} + \frac{1}{2}C_1x^2 + C_2x + C_3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_n(x) = \frac{1}{a^n}e^{ax} + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_{n-1}x + k_n,$$

in which k_1, k_2, $\cdots$, k_n involve the constants of integration.

Ex. 2. A body falls with a constant acceleration $g = 32.2$ ft./sec.². If it starts from rest, through what distance will it fall in 10 seconds?

We have here $$\frac{d^2s}{dt^2} = 32.2, \tag{1}$$

$$\frac{ds}{dt} = \int \frac{d^2s}{dt^2}\,dt = \int 32.2\,dt = 32.2\,t + C_1, \tag{2}$$

$$s = \int \frac{ds}{dt}\,dt = \int (32.2\,t + C_1)\,dt = 16.1\,t^2 + C_1 t + C_2. \tag{3}$$

We may now determine the constants C_1, C_2, from the initial conditions of the problem. Since the body falls from rest, we have for $t = 0$,

$$s = 0, \quad \frac{ds}{dt} = 0.$$

Hence for $t = 0$, we have from (2) and (3)

$$C_1 = 0, \quad C_2 = 0.$$

Therefore, for the time t, the space passed over is given by

$$s = 16.1\,t^2.$$

For $t = 10$ seconds, $s = 1610$ feet.

EXERCISES

Find four successive integrals of the following functions.

1. $y = \sin ax$. **2.** $y = x^2 - 1$. **3.** $y = \frac{1}{x^4}$.

4. Find a curve that passes through the points $(0, 0)$ and (m, n) and for which the second derivative $\frac{d^2y}{dx^2}$ at any point is k times the abscissa at that point.

5. In the theory of flexure of beams the following equation occurs:

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[M + Rx - \frac{wx^2}{2}\right],$$

E, I, M, R, and w being constants. Derive an expression for y and determine the constants of integration C_1 and C_2 from the conditions $y = 0$, when $x = 0$, and $y = 0$ when $x = l$.

6. A point has an acceleration expressed by the equation $a = - r\omega^2 \cos \omega t$, where r and ω are constants. Derive expressions for the velocity and the distance traveled.

We have $$\frac{d^2s}{dt^2} = - r\omega^2 \cos \omega t.$$

Hence, $$v = \frac{ds}{dt} = \int \frac{d^2s}{dt^2}\,dt = - r\omega^2 \int \cos \omega t\,dt$$
$$= - r\omega \sin \omega t + C',$$

and
$$s=\int\frac{ds}{dt}dt=-r\omega\int\sin\omega t\,dt+\int C'dt$$
$$=r\cos\omega t+C't+C'',$$
which is the law of simple harmonic motion.

7. In Ex. 6, determine the constants of integration from the following initial conditions: $s=r$ and $v=0$, when $t=0$.

84. Maxima and minima. In Chapter III was discussed the relation of the derivative to increasing and decreasing functions and to turning points of a curve. At a turning point, the function has the largest or the smallest value of all values in that neighborhood; and we say that it has a maximum or a minimum value at such a point. A maximum value of a function may therefore be defined as follows:

A function $f(x)$ is said to have a **maximum** *for $x=a$, if $f(a)$ is greater than the values of $f(x)$ for x just preceding and just following $x=a$.* Expressed symbolically, $f(x)$ has a maximum for $x=a$, if
$$f(a)>f(a\pm h),$$
where h is positive and takes all values in the immediate neighborhood of zero.

Likewise, *a function $f(x)$ is said to have a* **minimum** *for $x=a$, if $f(a)$ is less than the values of $f(x)$ for x just preceding and just following $x=a$;* in other words, if
$$f(a)<f(a\pm h),$$
for all values of h in the neighborhood of zero.

A function may have several maxima and several minima in any given interval. We shall limit the present discussion to functions having only a finite number of maxima and minima in a finite interval.

The analytic conditions for a maximum or a minimum are easily deduced when the function has a derivative. If $f(a)$ is a maximum, $f(x)$ must increase with x just before $x=a$ and decrease as x increases just beyond. As shown in Art. 34, the derived function $f'(x)$ must then be positive for values of x just preceding $x=a$ and negative for all values just following. Similarly, if $f(a)$ is a minimum, $f'(x)$ must be negative for values of x

just preceding $x = a$ and positive for values just following. We may combine these statements in the following theorem:

THEOREM I. *$f(a)$ is a maximum or a minimum of the function $f(x)$, according as $f'(x)$ changes from positive to negative or from negative to positive values as x, for increasing values, passes through the value $x = a$. If $f'(x)$ does not change sign as x passes through $x = a$, then $f(x)$ has neither a maximum nor a minimum at $x = a$.*

To apply this test, substitute $(a - h)$ and $(a + h)$ for x in $f'(x)$. In case $f(a)$ is a maximum, $f'(a - h)$ remains positive and $f'(a + h)$ negative however small the value of h is taken; if $f(a)$ is a minimum, then $f'(a - h)$ remains negative and $f'(a + h)$ positive for arbitrarily small values of h.

If the maximum or minimum occurs at an ordinary point of the curve, the tangent at the point is parallel to the X-axis, and consequently the first derivative for this value of x must vanish; that is, $f'(a) = 0$. We may determine whether $f'(x)$ passes from positive to negative values or *vice versa* by examining the character of *its* derivative in the neighborhood of $x = a$. In case of a maximum, $f'(x)$ is first positive, then zero at $x = a$, and finally negative for values following a. In other words, as $f(x)$ passes through the maximum value, $f'(x)$ decreases. Hence, by Art. 34, $f''(x)$ must be either negative or zero for $x = a$. On the other hand, as $f(x)$ passes through a minimum, $f'(x)$ is first negative, then zero, and finally positive; that is, $f'(x)$ is an increasing function at the point $x = a$. Hence $f''(x)$ must be either positive or zero for $x = a$. For the case in which $f'(a)$ becomes zero while $f''(a)$ remains different from zero, we may therefore express the condition for a maximum or a minimum as follows:

THEOREM II. *If $f'(a) = 0$, then $f(a)$ is a maximum or a minimum according as $f''(a)$ is negative or positive.*

This theorem affords an easy method of determining the maximum or the minimum points of a function. We equate the first derivative to zero and solve for the real values of the variable. If upon substituting these values for the variable in the second derivative we obtain a negative number, we have a maximum; if we obtain a positive number, we have a minimum.

It may occur that both $f'(x)$ and $f''(x)$ vanish for the same value a. In this case the last method of testing the given function for maxima and minima fails, and we must apply the first method. Theorem I must also be applied when $f'(x)$ changes sign by passing through infinity, as x passes through $x=a$, or by having a finite discontinuity, as in Fig. 26, (a) and (b) respectively.

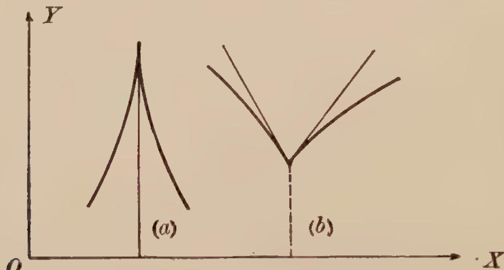

FIG. 26.

The special cases in which higher derivatives vanish simultaneously with $f'(x)$ will be considered in a later section, where a second proof purely analytic in its nature will be given.

Ex. 1. Examine the function $(x+2)^2(x-1)^3$ for maximum and minimum values.

$$f(x)=(x+2)^2(x-1)^3,$$

$$f'(x)=2(x+2)(x-1)^3+3(x+2)^2(x-1)^2$$
$$=5(x+2)(x-1)^2(x+\tfrac{4}{5}).$$

From $$(x+2)(x-1)^2(5x+4)=0,$$

we have the roots $x=-2,\ +1,\ -\frac{4}{5}$. These are critical values to be examined.

We may test the value $x=-2$ by examining the original function; thus

$$f(-2)=0,$$
$$f(-2+h)=h^2(-2+h-1)^3<0,$$
$$f(-2-h)=h^2(-2-h-1)^3<0.$$

Therefore, since the function is greater for $x=-2$ than for values of x just preceding or following it, it follows that for $x=-2$, the function has a maximum. The value $x=1$ we shall likewise examine by substituting $x=1$, $x=1+h$, and $x=1-h$ in the original function.

$$f(1)=0,\ f(1+h)=(1+h+2)^2h^3>0,$$
$$f(1-h)=-(1-h+2)^2h^3<0.$$

For this value, the function has neither a maximum nor a minimum, although $f'(x) = 0$. For the critical value $x = -\frac{4}{5}$, we may examine the derivative $f'(x)$.

For $x = -\frac{4}{5} - h$, $f'(x)$ is negative,

and for $x = -\frac{4}{5} + h$, $f'(x)$ is positive.

Since $f'(x)$ changes from negative to positive as x passes through the value $-\frac{4}{5}$, the function has a minimum for this value of x.

The graph of this function is shown in Fig. 27.

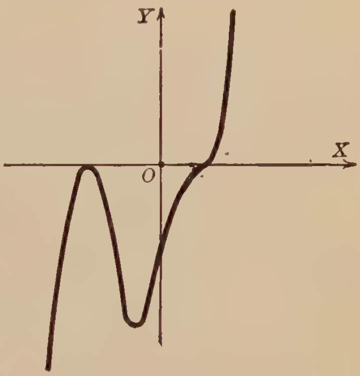

Fig. 27.

Ex. 2. Examine $x^3 - 4x^2 + 5x - 2$ for maxima and minima.

$$f(x) = x^3 - 4x^2 + 5x - 2,$$

$$f'(x) = 3x^2 - 8x + 5 = (3x-5)(x-1),$$

$$f''(x) = 6x - 8.$$

Putting $f'(x) = 0$, the critical values are seen to be $x = \frac{5}{3}$, $x = 1$. For $x = \frac{5}{3}$, $f''(x) = 6 \times \frac{5}{3} - 8 = 2$, and for $x = 1$, $f''(x) = -2$; hence, the function has a minimum for $x = \frac{5}{3}$ and a maximum for $x = 1$.

Ex. 3. Let $$f(x) = \frac{x}{\log x}.$$

Then $$f'(x) = \frac{\log x - 1}{(\log x)^2}.$$

Hence for $f'(x) = 0$, $\log x = 1$ or $x = e$ gives the critical value. For $x < e$, $\log x < 1$, and $f'(x)$ is negative, while for $x > e$, $\log x > 1$ and $f'(x)$ is positive; hence, $f(e)$ is a minimum value of $f(x)$.

Ex. 4. Examine $(x-4)^{\frac{2}{3}} + 5$ for maxima and minima.

Here $$f(x) = (x-4)^{\frac{2}{3}} + 5,$$

whence $$f'(x) = \frac{2}{3(x-4)^{\frac{1}{3}}}.$$

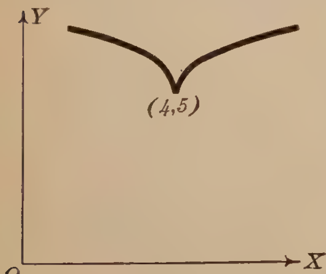

Fig. 28.

For $x = 4$, $f'(x) = \infty$; hence $x = 4$ is a critical value. For $x > 4$, $f'(x)$ is positive, while for $x < 4$, $f'(x)$ is negative. Hence as x increases, $f'(x)$ changes from negative to positive, and from Theorem I, $f(x)$ has a minimum at $x = 4$. See Fig. 28.

EXERCISES

Examine the following functions for maximum and minimum values.

1. $y = x^3 - 10x^2 + 30.$

2. $y = x^4 - 6x^2 + 10.$

3. $y = x(x^2 - 1).$

4. $y = x^2(x - 1).$

5. $y = \dfrac{x}{\sqrt{x-1}}.$

6. $y = \dfrac{x^2}{2} + \dfrac{1}{x}.$

7. $y = \dfrac{x+2}{x}\sqrt{1+x^2}.$

8. $y = e^x + e^{-x}.$

9. $y = xe^{-x}.$

10. $y = \sin x + \cos x,\ 0 \leqq x \leqq \dfrac{\pi}{2}.$

11. $y = 2\tan x + \sec^2 x.$

12. $y = (x-2)^{\frac{2}{3}} + 6.$

13. $y = x\sqrt{ax+b}.$

14. $y^2 = x^2 + x^3.$

15. $y = x^3 - xy.$

16. $y = x^{-1}\log x.$

17. $y = (x+2)^{\frac{2}{3}}(x-3)^2.$

18. $y = x - e^x.$

85. Applications of maxima and minima. The theory of maxima and minima has many important applications in geometry, mechanics, and physics, a few of which are given in the following problems. In some of these problems the function whose maximum or minimum value is required is given; in others, however, the function must be found from the conditions stated in the problem. Frequently the function derived will contain more than one variable; but in such cases, the conditions of the problem will furnish certain relations between the variables by means of which all but one can be eliminated. The function of the single variable thus obtained will be treated according to the methods given in the preceding section. A careful study of the following illustrative examples will give a grasp of the proper method of attack.

Ex. **1.** A rectangular channel, Fig. 29, carrying a given volume of water, is to be so proportioned as to have a minimum wetted perimeter (*i.e.* the part of the perimeter of the channel in contact with water). Determine the proportions of the channel.

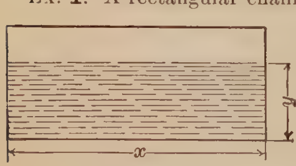

FIG. 29.

Let x = width of bottom, and y = height of water. The volume of water is proportional to the cross section; since this volume is to be constant, we have

$$xy = C. \tag{1}$$

The wetted perimeter p is given by the equation

$$p = x + 2y. \tag{2}$$

This is the function whose minimum value is required. It contains two variables x and y, but one may be eliminated by means of the first relation; thus,

$$y = \frac{C}{x},$$

whence
$$p = x + 2\frac{C}{x}. \tag{3}$$

Differentiating (3), we get
$$\frac{dp}{dx} = 1 - \frac{2C}{x^2}. \tag{4}$$

In order that p shall be a maximum or minimum, we must have

$$1 - \frac{2C}{x^2} = 0,$$

whence
$$x = \sqrt{2C},$$

and
$$y = \tfrac{1}{2}\sqrt{2C};$$

that is,
$$x = 2y. \tag{5}$$

To determine whether this ratio between bottom and side gives a maximum or minimum, we observe that $\frac{dp}{dx}$ is negative for $x^2 < 2C$ and positive for $x^2 > 2C$. This shows that p is a minimum for $x = 2y$.

Ex. **2.** The deflection of a rectangular beam of a fixed length under a given load varies inversely as the product of the breadth and the cube of the depth. From a log a inches in diameter, a beam is to be cut of such dimensions as to make the deflection a minimum.

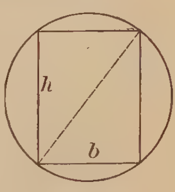

Fig. 30.

Denoting by z the deflection, and by b and h the breadth and depth, respectively, the equation

$$z = \frac{k}{bh^3} \tag{6}$$

expresses the law stated above.

There are two variables in this function, but one of them may be eliminated by means of a second equation derived from the geometry of the figure, viz.:

$$b^2 + h^2 = a^2. \tag{7}$$

Combining (6) and (7), we have

$$z = \frac{k}{h^3\sqrt{a^2 - h^2}}. \tag{8}$$

By differentiation of (8), we obtain

$$\frac{dz}{dh} = \frac{k(4h^2 - 3a^2)}{h^4(a^2 - h^2)^{\frac{3}{2}}}. \tag{9}$$

This derivative takes the value zero when $4h^2 - 3a^2 = 0$, that is, when

$$h = \frac{a}{2}\sqrt{3}.$$

The corresponding value of b is, from (7), $\frac{a}{2}$. The student may show that for these values z is a minimum, not a maximum.

EXERCISES

1. Divide a number a into two parts such that the sum of their squares shall be a minimum.

2. The range of a projectile is given by the expression $R = \frac{v_0^2 \sin 2\phi}{g}$, in which v_0 denotes the initial velocity and ϕ the angle which v_0 makes with the horizontal. For what angle ϕ is the maximum range obtained? What is the maximum range?

3. The efficiency of a screw as a mechanical device is given by the formula

$$E = \frac{h(1 - h\mu)}{h + \mu},$$

where the constant μ is the coefficient of friction, and h is the tangent of the pitch angle of the screw. Find the value of h for which the efficiency is a maximum.

4. It is desired to make an open-top box of greatest possible volume from a square piece of tin whose side is a, by cutting out equal squares from the corners and then folding up the tin to form the sides. What should be the length of the side of the square cut out?

5. A roofer wishes to make an open trapezoidal gutter of maximum capacity whose bottom and sides are each 4 inches wide and whose sides have the same slope. What should be the width across the top?

6. Determine the most economical proportions for a cylindrical tank with flat heads; that is, find the ratio of the length to the diameter.

7. Find the minimum distance from the point (4, 0) to the parabola $y^2 = 6x$.

8. Find the minimum distance from the point $(-6, 0)$ to the positive branch of the hyperbola $xy = 8$.

9. The radius of curvature of the hyperbola $xy = c^2$ at the point (x_1, y_1) is $\frac{(x_1^2 + y_1^2)^{\frac{3}{2}}}{2c^2}$ (see Art. 92). Show that this radius is a minimum at the point (c, c), and find the minimum value.

10. Determine the right circular cone of minimum volume that can be circumscribed about a given sphere.

11. Given a triangle one of whose vertices lies at the center of a circle of radius a. If two sides of the triangle are radii of the circle, show that $\frac{1}{2}a^2$ is the maximum area of the triangle.

Find the dimensions of the following inscribed figures that will give maximum area.

12. Rectangle in a circle of radius r.

13. Rectangle in ellipse of semiaxes a and b.

14. Isosceles triangle in circle of radius a.

Find the dimensions of the following inscribed solids in order that the volume shall be a maximum.

15. Cylinder in right circular cone.

16. Cone in a sphere of radius a.

17. Find the dimensions of a conical tent that for a given volume V will require the least material.

18. The power P developed by a Pelton water wheel is proportional to the speed v of the wheel buckets and also to the relative velocity $c-v$ with which the jet issuing from the nozzle with speed c strikes the buckets; that is, $P=kv(c-v)$, where k is a constant. With a given value of the jet speed c, determine the bucket speed v that will give maximum power.

19. The weight of hot gas passing up a chimney in a given time is given by the formula

$$W=\frac{k\sqrt{0.96\,T-T_1}}{T},$$

where T and T_1 denote respectively the absolute temperatures of the gas and the outside air, and T_1 is constant. Find the ratio $\frac{T}{T_1}$ for which W is a maximum.

20. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the dimensions of the beam of maximum strength that can be cut from a log 14 inches in diameter.

21. Find the beam of greatest stiffness that can be cut from a log 12 inches in diameter, knowing that the stiffness varies as the breadth and the cube of the depth.

22. A body of weight W is dragged along a horizontal plane by means of a force P whose line of action makes an angle θ with the plane. The magnitude of the force is given by the equation

$$P=\frac{\mu W}{\mu\sin\theta+\cos\theta},$$

in which μ denotes the coefficient of friction. Show that the pull is least when $\theta=\operatorname{arc}\tan\mu$.

MISCELLANEOUS EXERCISES

Find $\frac{d^2y}{dx^2}$ for the following.

1. $y = x^2 \sin x$.

2. $y = e^x \sin\left(x + \frac{\pi}{4}\right)$.

3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

4. $y = \frac{a^2 x}{\sqrt{x^2 - a^2}}$.

5. If $y = Ae^{ax} + Be^{-ax}$, show that $\frac{d^2y}{dx^2} - a^2 y = 0$.

6. If $s = C_1 \sin kt + C_2 \cos kt$, show that $\frac{d^2s}{dt^2} + k^2 s = 0$.

7. If u and v are functions of x, show that

$$D_x^n(uv) = uD_x^n v + nD_x^{n-1}vD_x u + \frac{n(n-1)}{2!}D_x^{n-2}vD_x^2 u + \cdots + vD_x^n u.$$

Observe that the coefficients and symbolic exponents follow the law of coefficients and exponents in the expansion of a binomial. This is *Leibnitz' theorem.* (Compare *Gibson's Calculus*, § 68.)

8. Using the result of Ex. 7, find

(*a*) $D_x^5 y$ when $y = e^x \cos x$;

(*b*) $D_x^4 y$ when $y = x^3 \log x$.

9. If $\frac{d^2y}{dx^2} = \frac{1}{EI}\left[M - \frac{wlx}{2} + \frac{wx^2}{2}\right]$, find $y = f(x)$ knowing that $\frac{dy}{dx} = 0$ when $x = 0$ and when $x = l$; also that $y = 0$ when $x = 0$.

10. Find curves for which at any point the second derivative $\frac{d^2y}{dx^2}$ has the constant value $2a$. Which of these curves passes through the origin? Which has the slope a at the point $(-2, 0)$?

11. Find three successive integrals of the functions

(*a*) $-\frac{a}{x^3} + x + 5$. (*b*) $e^{ax} - e^{-ax}$. (*c*) $\sin(kt + e)$.

12. Find expressions for the acceleration $\frac{d^2x}{dt^2}$ for the motions described by the equations

(*a*) $x = c_1 e^{-\alpha t} + c_2 e^{-\beta t}$;

(*b*) $x = (c_1 + c_2 t)e^{-\alpha t}$.

13. Find an expression for $\frac{d^2x}{dt^2}$ when the equation of motion is

$$x = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

14. From (*a*) of Ex. 12 deduce the relation

$$\frac{d^2x}{dt^2} + (\alpha + \beta)\frac{dx}{dt} + \alpha\beta x = 0.$$

15. The ideal efficiency of a certain type of water turbine is given by the equation $E = \frac{cu\sqrt{2gh + u^2} - u^2}{gh}$, where u denotes the variable peripheral speed of the wheel. Find the value of u for maximum efficiency, also the maximum efficiency.

16. Find maximum and minimum values (if such exist) of the functions

$$(a)\ \frac{x^2 - 2x + 5}{x^2 - 4x + 7}; \qquad (b)\ \sin\theta(1 + \cos\theta).$$

17. Find $\frac{d^2y}{dx^2}$ for the curve given by the parametric equations

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta).$$

18. Find $\frac{d^2y}{dx^2}$ for the conic section $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

19. Find the acceleration $\frac{d^2s}{dt^2}$ from the equation

$$s = L + r(1 - \cos\theta) - \sqrt{L^2 - r^2\sin^2\theta},$$

knowing that $\frac{d\theta}{dt}$ is a constant ω_0. This is the acceleration of the piston of the steam engine mechanism, L being the length of connecting rod, r that of the crank, and ω_0 the angular speed of the crank.

20. Show that the maximum and minimum values of an integral algebraic function occur alternately.

21. The specific heat of superheated steam is given by the equation $c = \alpha + \beta T + p\left(1 + \frac{a}{2}p\right)\frac{C}{T^{n+1}}$, where T denotes the absolute temperature, p the pressure, and α, β, a, C, and n are constants. If p is kept constant, show that c takes a minimum value for some temperature T_m and that this value is $\alpha + \frac{n+2}{n+1}\beta T_m$.

22. A rectilinear motion is such that the acceleration is given by the expression $a = e^t + e^{-t}$; derive expressions for velocity and distance. Show that if the particle starts from rest, the distance is numerically equal to the acceleration.

23. Find expressions for velocity and distance when the acceleration is given by $a = m - nk^2\cos kt$. Determine the constants of integration C_1 and C_2 by taking $v = 0$ and $s = 0$ when $t = 0$.

CHAPTER IX

CURVES

86. Concavity. In the present chapter we shall discuss some of the applications of differentiation to plane curves, and develop principles that will enable the student to trace a given curve and study its properties.

A curve is said to be **concave upward** at any point if an arc of the curve containing the point lies above the tangent to the curve at the point. It is said to be **concave downward** if the tangent lies above the arc. Thus the curve shown in Fig. 31 is concave downward between the points A and B, and it is concave upward from B to C.

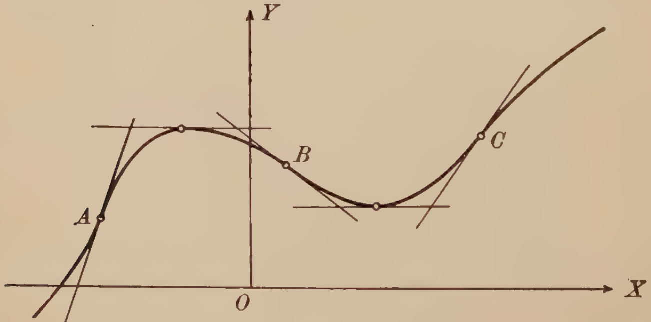

FIG. 31.

The condition for concavity upward or downward can be expressed in terms of the derivatives of the function represented by the curve. When the arc of the curve $y = f(x)$ lies below the tangent, as between the points A and B, $\tan \phi$, and consequently $\frac{dy}{dx}$, decreases as x increases. On the other hand, when the arc lies

above the tangent, $\frac{dy}{dx}$ increases with x. We have therefore the general statement:

A given curve is concave upward or downward according as the derivative $\frac{dy}{dx}$ *is an increasing or a decreasing function.*

When the derivative $\frac{dy}{dx}$ is decreasing, its derivative, namely the second derivative, $\frac{d^2y}{dx^2}$, must be either negative or zero; if $\frac{dy}{dx}$ is increasing, then the second derivative is either positive or zero. Therefore, if $\frac{d^2y}{dx^2}$ is different from zero, the preceding statement is equivalent to the following:

A given curve is concave upward if the second derivative is positive, and concave downward if the second derivative is negative.

Ex. Given the curve $y = x^3 - x^2 + 6$, investigate its concavity.

The second derivative, $\frac{d^2y}{dx^2} = 6x - 2$, is positive for values of $x > \frac{1}{3}$, and is negative for values of $x < \frac{1}{3}$. Hence, to the left of $x = \frac{1}{3}$ the curve is concave downward, and to the right of that point it is concave upward.

EXERCISES

Test the following curves for concavity upward or downward.

1. $y = x^3 - 2x^2 + 5$. **2.** $y = \sec x$. **3.** $y = \frac{x(1-x)}{1+x}$.

4. $y = a\sqrt{x-a}$. **5.** $y = \frac{1}{2}(e^x + e^{-x})$.

6. Show that a curve is concave or convex toward the X-axis according as $y\frac{d^2y}{dx^2}$ is negative or positive.

7. Show that the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is everywhere concave toward the X-axis.

8. Test the curves $y = e^x$ and $y = \log x$ for concavity.

9. Test the expansion curves given by the general equation $p^m v^n = C$ for concavity.

10. Show that the curve $ay^2 = x^3$ has two branches, each of which is convex to the X-axis.

87. Points of inflexion. A point at which a curve having a continuous slope ceases to be concave upward and becomes concave downward, or *vice versa*, is called **a point of inflexion** of the given curve. The curve shown in Fig. 31 has points of inflexion at A, B, C. It is evident that at such points the curve crosses its tangent.

The definition suggests the analytic condition for a point of inflexion. As has been shown, so long as the curve $y = f(x)$ is concave upward the first derivative increases as x increases, and when it is concave downward the derivative decreases as x increases. It follows then that, as x passes through a value for which the curve has a point of inflexion, the first derivative passes from an increasing to a decreasing function, or *vice versa*. See Fig. 10. This is precisely the condition that the derived function shall have a maximum or minimum; and to determine the points of inflexion of the curve, we need only to examine the derived function for maxima and minima. It is therefore a necessary and sufficient condition for a point of inflexion that the second derivative shall change sign as the independent variable passes through the critical value. This change occurs when the second derivative passes through zero or through infinity; hence the coördinates of the points of inflexion of the curve may in general be found by solving the equations

$$f''(x) = 0, \quad f''(x) = \infty,$$

and determining whether $f''(x)$ changes sign as x passes through the values thus obtained.

Ex. Given the curve whose equation is $y = 5x^3 + 6x - 7$. We have

$$f''(x) = \frac{d^2y}{dx^2} = 30x,$$

which vanishes for $x = 0$. Moreover, $f''(x)$ changes sign as x passes from negative to positive values. Hence the curve has a point of inflexion at $x = 0$.

EXERCISES

Test the following curves for points of inflexion.

1. $y = 2x^3 - 3x^2 + 4x - 6$. **2.** $y = 3x^4 + 4x^2 - x + 10$.

3. $(y-3)^2 = x + 5$. **4.** $y = \sin x$. **5.** $y = \cot x$.

6. $y = x^2 - e^x$. 7. $y = e^x - e^{-x}$. 8. $y = \text{arc cot } x$

9. $y = e^{-x^2}$. 10. $y = \dfrac{x}{x^2+3}$. 11. $y = \dfrac{a}{1+bx^2}$.

12. Given a continuous curve. Draw a tangent to this curve, and through some fixed point draw a straight line which shall so change as to remain parallel to the tangent as the point of tangency changes. Show that as the point of tangency passes through a point of inflexion, the line through the fixed point changes the direction of its rotation. For this reason, the tangent at a point of inflexion is sometimes called a stationary tangent. Express in terms of the derivatives of the given function the condition for a stationary tangent.

13. Find the equation of a curve which has a point of inflexion at the point (0, 3) such that the inflexional tangent makes an angle of 45° with the X-axis. How many such curves can be found?

14. Show that the curve $y = \dfrac{1-x}{1+x^2}$ has three points of inflexion and that these points lie in a straight line.

88. Asymptotes, rectangular coördinates. An **asymptote** to a plane curve is a straight line, lying partly within the finite region, which is the limiting position of a tangent to the curve as the point of tangency recedes indefinitely along an infinite branch.

Two conditions are necessary for an asymptote: (1) The curve must have at least one infinite branch. Thus an ellipse having no infinite branches cannot have an asymptote. (2) The limiting position of the tangent must lie partly within the finite portion of the plane. For example, the tangent to a parabola at infinity is not an asymptote since it lies wholly at infinity.

There are two general methods of determining the asymptotes to a curve whose equation is given in rectangular coördinates.

Method of limiting intercepts. The equation of the tangent at the point (x_1, y_1) being

$$y - y_1 = f'(x_1)(x - x_1), \qquad \text{(Art. 38)}$$

the intercepts of the tangent on the coördinate axes are respectively:

$$\text{Intercept on } X\text{-axis} = x_1 - \frac{y_1}{f'(x_1)}. \qquad (1)$$

$$\text{Intercept on } Y\text{-axis} = y_1 - x_1 f'(x_1). \qquad (2)$$

If one or both of the intercepts has a finite value for $x_1 = \infty$ or $y_1 = \infty$, the infinite branch has an asymptote, and the equation of the asymptote may be found from the two intercepts or from one intercept and the limiting value of $f'(x_1)$. The following example will illustrate this method.

Ex. 1. Examine the curve whose equation is $y^3 = x^3 - 3x^2$ for asymptotes. Differentiating, we obtain

$$3y^2\frac{dy}{dx} = 3x^2 - 6x,$$

whence

$$f'(x_1) = \frac{x_1^2 - 2x_1}{y_1^2}.$$

From (1) and (2) we have

$$X\text{-intercept} = x_1 - \frac{y_1^3}{x_1^2 - 2x_1} = \frac{x_1^2}{x_1^2 - 2x_1},$$

$$Y\text{-intercept} = y_1 - \frac{x_1^3 - 2x_1^2}{y_1^2} = -\frac{x_1^2}{(x_1^3 - 3x_1^2)^{\frac{2}{3}}}.$$

For $x_1 = \infty$, these intercepts are respectively 1 and -1. Remembering that the equation of a line in terms of its intercepts a and b is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

we have as the equation of the asymptote,

$$x - y = 1.$$

Method of substitution. Let $f(x, y) = 0$ be the equation of the given curve, and assume the equation of the tangent to be

$$y = mx + b. \tag{3}$$

Combining these equations, we obtain the equation

$$f(x, mx + b) = 0. \tag{4}$$

Now a tangent to a curve may be regarded as the limiting position of a secant line as the two points in which this line intersects the curve are made to approach coincidence. Hence if the line $y = mx + b$ is tangent to the curve $f(x, y) = 0$, equation (4) must have equal roots, and if the point of tangency recedes indefinitely these roots become infinite. The condition that two of the roots of (4) shall be infinite is that the coefficients of the two highest powers of x in (4) shall vanish.* If, therefore, we

* Rietz and Crathorne's *College Algebra*, Art. 111.

equate these coefficients to zero, we can determine the values of m and b, and consequently the equation of the asymptote.

Ex. 2. Examine for asymptotes the curve whose equation is

$$x^4 - x^3y + xy - y^3 = 0.$$

Substituting $y = mx + b$, the resulting equation is

$$x^4(1-m) + x^3(-m^3 - b) + \cdots - b^3 = 0.$$

Equating the coefficients of x^4 and x^3 to 0, we have

$$1 - m = 0, \quad m^3 + b = 0,$$

whence $$m = 1, \quad \text{and} \quad b = -1.$$

Substituting these values in (3), we have as the equation of the asymptote

$$y = x - 1.$$

If in equation (4) the coefficients of x^n and x^{n-1} contain both m and b, the terms of degree lower than the $(n-1)$th do not in general affect the determination of the asymptotes. However, if the coefficient of x^{n-1} is zero, or if the value of m obtained by equating the coefficient of x^n to zero is such as to cause that of x^{n-1} to vanish also, we must equate the coefficient of the next lower degree to zero in order to have two equations from which m and b can be uniquely determined. This coefficient, in general, will be of the second degree in b, and hence we have in this case two parallel asymptotes, since for each value of m there are two values of b.

Method of inspection. In case the asymptotes are parallel to one of the coördinate axes, we can often determine such asymptotes by inspection. For this purpose the equation of the curve is written in descending powers of x or of y. It will then have either of the following forms:

$$ax^n + (by + c)x^{n-1} + \cdots + t = 0, \tag{5}$$

$$\alpha y^n + (\beta x + \gamma)y^{n-1} + \cdots + \tau = 0. \tag{6}$$

If both a and $by + c$ vanish, then two roots of (5) become infinite, and the line

$$by + c = 0 \tag{7}$$

is an asymptote. Similarly, if in (6) both α and $\beta x + \gamma$ vanish, the line

$$\beta x + \gamma = 0 \tag{8}$$

is an asymptote to the given curve. Consequently, if the given equation, when arranged in descending powers of x (or y), does

not have the term x^n (or y^n), then the coefficient of x^{n-1} (or y^{n-1}) equated to zero gives an asymptote to the given curve.

If in (5), a, b, and c all vanish, then the coefficient of x^{n-2}, which is in general a quadratic in y, will, when equated to zero, determine two asymptotes, real or imaginary. A similar statement applies to (6).

Ex. **3**. The equation $4x^3 + 2x^2y - 6xy^2 - x^2 + 3y^2 - 1 = 0$ is a cubic in which y^3 is absent. Hence arranged in descending powers of y, it takes the form

$$0y^3 - (6x - 3)y^2 + (2x^2)y + (4x^3 - x^2 - 1) = 0.$$

If we make $6x - 3 = 0$, the coefficients of y^3 and y^2 both vanish. Hence the line $6x - 3 = 0$, or $x = \frac{1}{2}$ is an asymptote. For $x = \frac{1}{2}$, we have also $y = -\frac{3}{2}$, so that the asymptote cuts the curve at the point $(\frac{1}{2}, -\frac{3}{2})$.

Ex. **4**. The equation $x^2y^2 - 4xy^2 + 3x^2y - 4x^2 - 5 = 0$ is of the fourth degree, but lacks the terms in x^4, x^3, y^4, and y^3. It may be arranged in the forms

$$0x^4 + 0x^3 + (y^2 - 4)x^2 - (4y^2 - 3y)x - 5 = 0,$$
$$0y^4 + 0y^3 + (x^2 - 4x)y^2 + (3x)y - (4x^2 + 5) = 0.$$

From the first we have two asymptotes,

$$y + 2 = 0, \quad \text{and} \quad y - 2 = 0,$$

determined by placing the coefficient of x^2 equal to zero. From the second, the asymptotes $x = 0$, $x - 4 = 0$ are obtained.

The methods given are sufficient to determine all the asymptotes to an algebraic curve. The second method will be found most convenient in determining asymptotes that make an oblique angle with the axes of coördinates, and the third in finding asymptotes parallel to the axes. If the equation of the curve involves a transcendental function, the first method may be used.

EXERCISES

Find the asymptotes of the following curves:

1. $y^3 = x^2(x - b)$.

2. $y^2 = x^2\left(\dfrac{x + a}{x - a}\right)$.

3. $y^2 = \dfrac{x^3}{2a - x}$.

4. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

5. $y^2(x - 2) = x^2$.

6. $y^3 - y^2 - x^2 + x^3 = 0$.

7. $x^2y^2 - c^2(x^2 + y^2) = 0$.

8. $x^3 + xy^2 - y^2 = 0$.

9. $x^3 - xy^2 + ay^2 = 0.$

10. $y = \dfrac{8a^3}{x^2 + 4a^2}.$

11. $y^3 = 6x^2 + x^3.$

12. $xy = 1.$

13. $x^3 - y^3 - x^2 + 2y^2 = 0.$

14. $x^2y + xy^2 = 8.$

15. $y + xy - x^3 = 0.$

16. $x^2y^2 - x^3 - y^3 = 0.$

89. Singular points. Certain points of a plane curve may have peculiarities not possessed by other points. Thus there may be a **multiple point**, where two or more branches of the curve intersect, Fig. 32 (*a*) and (*b*); a **tacnode**, where two branches of the curve come in contact and have a common tangent at the point, Fig.

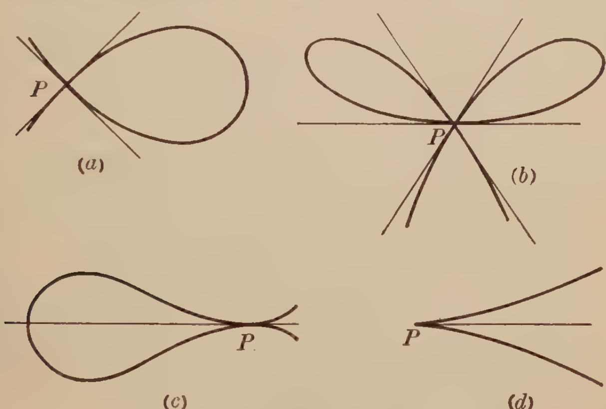

FIG. 32.

32 (*c*); a **cusp**, where the two branches terminate at the point of contact and have a common tangent, Fig. 32 (*d*); or a **conjugate point**, the coördinates of which satisfy the equation of the curve yet through which no branches of the curve pass. All such points are called **singular points**. At a singular point the derivative $\dfrac{dy}{dx}$ has two or more values, real or imaginary, equal or distinct. Geometrically this means that at a singular point the curve has two or more tangents, though some of these may be coincident, and some or all may be imaginary. The character of the curve at a singular point depends therefore upon the values of $\dfrac{dy}{dx}$ at that point. A general method of evaluating

$\frac{dy}{dx}$ at singular points cannot be given until later; fortunately, however, simple special methods are sufficient for the cases that ordinarily arise. The following examples are illustrative.

Ex. 1. Examine the curve $y^2 = x^2 - x^4$ for singular points.

Forming the derivative, we have $\frac{dy}{dx} = \frac{2x - 4x^3}{2y} = \frac{2x - 4x^3}{2x\sqrt{1-x^2}}$. For $0 < x < 1$, or for $-1 < x < 0$, every value of x gives two distinct values of y and two corresponding determinate values of $\frac{dy}{dx}$. For $x = 0$, there is one value of y, namely $y = 0$, showing that two branches of the curve pass through the origin. But for $x = 0$, $\frac{dy}{dx}$ takes the indeterminate form $\frac{0}{0}$ and this must be evaluated. Using the method of Art. 15, we have

$$\underset{x \doteq 0}{L} \frac{2x - 4x^3}{2x\sqrt{1-x^2}} = \underset{x \doteq 0}{L} \frac{1 - 2x^2}{\sqrt{1-x^2}} = \pm 1.$$

Hence at the origin $\frac{dy}{dx} = \pm 1$ and the tangents to the two branches are respectively $y = x$ and $y = -x$. Evidently the curve lies between the limits $x = 1$ and $x = -1$. The student may draw the curve.

In relatively few cases can the indeterminate expression for $\frac{dy}{dx}$ be evaluated by simple methods as in Ex. 1. Generally the method shown in the following example is effective and is easily applied.

Ex. 2. Examine for singular points the curve given by the equation $x^4 - 4x^2y + y^3 = 0$.

In this case more than one branch of the curve passes through the origin; therefore the origin is a point to be examined. Substitute $y = mx$ in the given equation. The result is an equation in x,

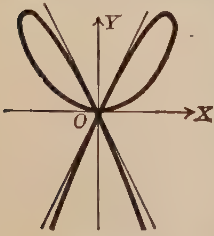

FIG. 33.

$$x^3(x - 4m + m^3) = 0, \tag{1}$$

which has three roots each equal to zero. Hence there are three branches passing through the origin, that is, the origin is a **triple point**. From (1) we have also the equation

$$x - 4m + m^3 = 0, \tag{2}$$

which gives the relation between the slope m of a secant line $y = mx$ that cuts the curve at the origin and in a second point whose abscissa is x. Since (2) is a cubic in m there are three such lines. As x approaches zero the

points of intersection approach coincidence and each secant line approaches as a limiting position a tangent at the origin. Therefore putting $x = 0$ in (2) and solving the resulting equation $m^3 - 4m = 0$, we get in the three roots $m = 0$, 2, and -2 the slopes of the three tangents to the three branches passing through the origin. The curve is shown in Fig. 33.

Ex. **3.** Examine for singular points the curve $ay = (x - b)^{\frac{5}{2}}$.

In general y has two values for any value of $x > b$, but for $x = b$, y has the single value 0; hence the point $(b, 0)$ is to be examined. For the sake of convenience let the origin be shifted to the point $(b, 0)$. Substituting $x - b = x$ and squaring to remove the radical, we have $a^2y^2 - x'^5 = 0$. Letting $y = mx'$, we obtain the equation in x',

$$x'^2(a^2m^2 - x'^3) = 0,$$

which has two zero roots, showing that two branches pass through the origin. From the equation $a^2m^2 - x'^3 = 0$, we have for $x = 0$, $a^2m^2 = 0$, an equation in m with equal zero roots. It follows that the branches have a common tangent $y = 0$. For negative values of x', that is, for $x < b$, y takes imaginary values; hence the two branches terminate at the point of contact, and this point $(b, 0)$ is therefore a cusp, Fig. 34.

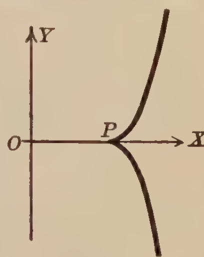

FIG. 34.

Ex. **4.** Examine the curve $x^3 - 3x^2 - 3y^2 + y^3 = 0$.

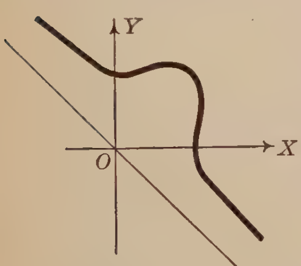

FIG. 35.

The origin is evidently a point to be examined. Putting $y = mx$, the result is

$$x^3 - 3x^2 - 3m^2x^2 + m^3x^3 = 0,$$

or $$x^2(x - 3 - 3m^2 + m^3x) = 0.$$

It follows that two branches (real or imaginary) pass through the origin. To determine the tangents at the origin, we place $x = 0$ in the equation $x - 3 - 3m^2 + m^3x = 0$. The result is $3m^2 + 3 = 0$, from which $m = \pm\sqrt{-1}$. Since the tangents are imaginary no real branches pass through the origin and therefore the origin is a conjugate point. The curve is shown in Fig. 35.

EXERCISES

Examine the following curves for singular points.

1. $x^4 - 2axy^2 + 2ay^3 = 0$.

2. $y^2 = \dfrac{x^3}{2a - x}$.

3. $y = \dfrac{x^2}{\sqrt{x^2 - a^2}}$.

4. $a^4y^2 = a^2x^4 - x^6$.

5. $x(x^2+y^2)+a(x^2-y^2)=0$. 6. $x^3-3axy+y^3=0$.

7. Show that the lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$ has a double point at the origin, and find the tangents at this point.

8. Show that the curve $y^2=x^3-4x^2$ has a conjugate point at the origin.

9. Show that the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ has a cusp at each of the points $(a, 0)$, $(-a, 0)$, $(0, a)$, and $(0, -a)$.

10. Find the equation of a curve that has a double point at the origin with tangents $y=x$, and $y=2x$.

11. Examine the curve $y-b=(x-a)\sqrt{x}$ for singular points.

12. The curve $y^2=ax^2+bx^3$ has a double point at the origin. Find the tangents at the origin by evaluating $\frac{dy}{dx}$ for $x=0$.

90. Curve tracing. It is frequently desirable to determine the general form of the graph of a given function. The direct method of tracing a curve is to determine simultaneous values of x and y from the given equation $y=f(x)$ and plot the points thus found. However, by a careful study of the derivatives of the function, the labor of this direct method can be largely avoided, and the general course of the graph, which is all that is required, can be found. In tracing a curve the student should proceed somewhat as follows:

1. Examine the equation for symmetry. If only even powers of x appear, the curve is symmetrical with respect to the Y-axis; if only even powers of y appear, it is symmetrical with respect to the X-axis; while if the equation remains unchanged when x and y are replaced by $-x$ and $-y$ respectively, the curve is symmetrical with respect to the origin.

2. Find the points in which the curve crosses the X- and Y-axes.

3. Find the *finite* values of x (or of y) for which y (or x) becomes infinite.

4. Find the slope of the curve by means of the first derivative and note the turning points.

5. By an examination of the first or second derivative determine whether the curve is concave upward or downward, and find the points of inflexion.

6. Examine the curve for asymptotes.

7. In some cases it may be advisable to examine the curve for singular points.

Having determined these characteristics of the curve, it is easy in most cases to sketch the graph. If a minute study of the form of the curve at any particular point is desired, it is well to transform the equation so that the point in question becomes the origin; then we need only consider the nature of the given curve in the neighborhood of the origin. In determining the nature of the curve in the neighborhood of the origin, the terms of the lowest degree in x and y are in general most important, since for small values of these variables the terms of higher order are small in comparison. Likewise the nature of the curve for very large values of the variables is determined by considering only the terms of highest order in the variables.

In the following examples we shall consider a few of the simpler curves, which are frequently referred to in subsequent chapters.

Ex. **1.** *The curves* $y = ax^n$. It is assumed that n is positive and either an integer or a fraction. For $n = 2$ and for $n = \frac{1}{2}$, the curves are ordinary parabolas. If $n = 3$, we have the cubical parabola $y = ax^3$. An examination of this equation discloses the following: (a) The curve passes through the origin. (b) For positive values of x, y is positive, and for negative values of x, y is negative; hence the curve lies wholly in the first and third quadrants, and is symmetrical with respect to the origin. (c) $\dfrac{dy}{dx} = 3\,ax^2$; hence the slope is always positive and increases as the numerical value of x increases. At the origin the slope is zero. (d) $\dfrac{d^2y}{dx^2} = 6\,ax$, hence to the right of the Y-axis the curve is concave upward, and to the left of the Y-axis it is concave downward. (e) At $x=0$, $\dfrac{d^2y}{dx^2}$ changes sign; hence the origin is a point of inflexion. From these conclusions it is evident that the curve has the general form shown in Fig. 36.

If $n = \frac{3}{2}$, the equation becomes $y^2 = a^2x^3$, and the curve is the semicubical parabola. From the equation we get

$$\frac{dy}{dx} = \frac{3}{2}\,a\sqrt{x}, \quad \frac{d^2y}{dx^2} = \frac{3}{4}\,\frac{a}{\sqrt{x}}.$$

It is readily seen that the curve is symmetrical with respect to the X-axis and lies wholly to the right of the Y-axis. At the origin the slope is zero, and the two branches of the curve have the X-axis as a common tangent;

the origin is therefore a cusp. There is no point of inflexion. Hence the curve has the general form shown in Fig. 37.

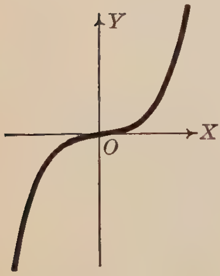

FIG. 36.

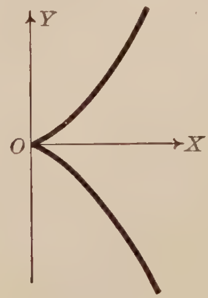

FIG. 37.

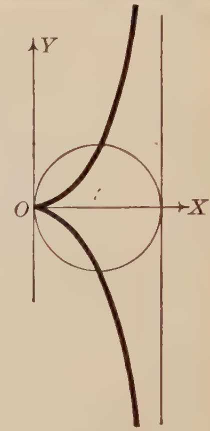

FIG. 38.

Ex. 2. *The cissoid* $y^2 = \dfrac{x^3}{2a - x}$. The curve is symmetrical with respect to the X-axis. For $x = 2a$, y becomes infinite; and for $x > 2a$ or $x < 0$, y is imaginary; hence the curve lies wholly between the Y-axis and the line $x = 2a$, which is an asymptote. For $x = 0$, $\dfrac{dy}{dx} = 0$, showing that the X-axis is a tangent to both branches at the origin. An investigation of the second derivative shows that there is no point of inflexion and that both branches are convex towards the X-axis. The curve has therefore the general form shown in Fig. 38.

Ex. 3. *The witch* $y = \dfrac{8a^3}{x^2 + 4a^2}$. The curve is symmetrical with respect to the Y-axis. y has a maximum value $2a$ for $x = 0$, and there are points

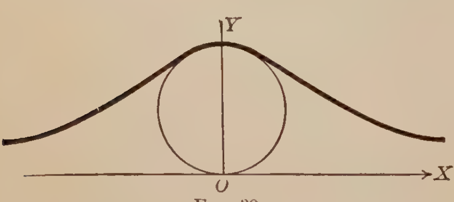

FIG. 39.

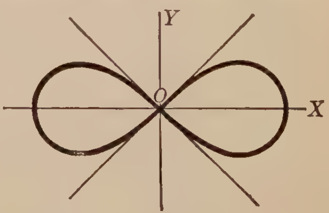

FIG. 40.

of inflexion at $x = \pm 2a\sqrt{3}$. The X-axis is an asymptote. The curve is shown in Fig. 39.

Ex. 4. *The lemniscate* $\rho^2 = a^2 \cos 2\theta$,

or $$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

The curve is symmetrical with respect to both axes, and crosses the X-axis at $x = \pm a$. For values of θ from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$ and from $\frac{5}{4}\pi$ to $\frac{7}{4}\pi$, ρ is imaginary. The origin is a double point, the tangents having the slopes -1 and 1, respectively. The curve has the form shown in Fig. 40.

Ex. 5. *The catenary* $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ is the curve assumed by a flexible cord of uniform weight suspended trom two fixed points. The curve is symmetrical about the Y-axis, and cuts the Y-axis at a distance a above the origin. The second derivative is positive for all values of x, hence the curve is concave upwards and has no point of inflexion. See Fig. 41.

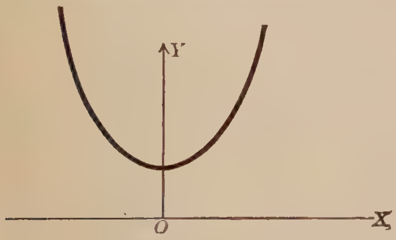

Fig. 41.

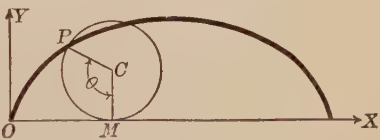

Fig. 42.

Ex. 6. *The cycloid* is the curve described by a point on the circumference of a circle which rolls on a straight line. Let a denote the radius of the circle, and θ the angle PCM, Fig. 42, subtended by the arc $PM (= OM)$. Then we have for the coördinates of P, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. From these equations, we readily obtain

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}, \quad \frac{d^2y}{dx^2} = -\frac{1}{a(1 - \cos\theta)^2}.$$

The curve has a turning point at $x = \pi a$, and since $\frac{d^2y}{dx^2}$ is always negative, the curve is everywhere concave downward.

Ex. 7. *The astroid*, or hypocycloid of four cusps, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is a curve described by a point on the circumference of a circle of radius $\frac{1}{4}a$ rolling within the circumference of a circle of radius a. See Fig. 43. The student may find the derivatives and by their aid study the curve for slope, concavity, etc.

Ex. 8. *The cardioid* $\rho = 2a(1 - \cos\theta)$. The curve is closed, and ρ is finite for all values of θ. Since $\cos\theta = \cos(-\theta)$, the curve is symmetrical with respect to the initial line OX. From the given equation, we get

$$\tan\psi = \rho\frac{d\theta}{d\rho} = \frac{1 - \cos\theta}{\sin\theta} = \tan\tfrac{1}{2}\theta, \qquad \text{(Art. 40)}$$

whence $$\psi = \tfrac{1}{2}\theta.$$

For $\theta = 0$, $\psi = 0$, that is, OX is a tangent at the origin; for $\theta = \frac{1}{2}\pi$, $\psi = \frac{1}{4}\pi$, and for $\theta = \pi$, $\psi = \frac{1}{2}\pi$. For the limits of ρ we have $\rho = 0$ when $\theta = 0$, $\rho = 4a$ when $\theta = \pi$. See Fig. 44.

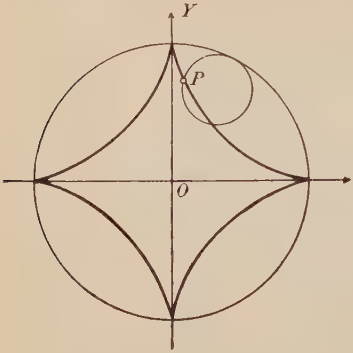

FIG. 43.

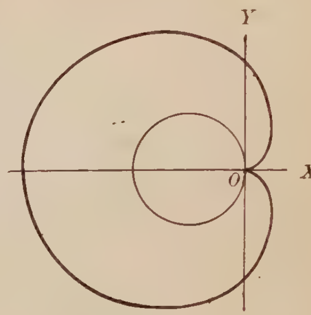

FIG. 44.

EXERCISES

The student should trace the following curves carefully and preserve the graphs for future reference.

1. The group of curves $x^m y = C$ (m positive). These are the curves which represent the laws of expanding gases.

2. The curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. Show that this curve is the ordinary parabola.

3. The folium of Descartes, $x^3 + y^3 - 3axy = 0$.

4. The exponential curve $y = e^x$.

5. The logarithmic curve $y = \log x$.

6. The logarithmic spiral $\rho = e^{a\theta}$.

7. The spiral of Archimedes, $\rho = a\theta$.

8. The curve $\rho = a \sec^2 \dfrac{\theta}{2}$.

9. The parabolic spiral $\rho^2 = a^2\theta$.

10. Rankine's equation for columns, $y = \dfrac{a}{1 + bx^2}$.

11. The probability curve $y = ke^{-ax^2}$, $a > 0$. Find turning points and points of inflexion.

12. The lituus $\rho^2\theta = a^2$.

13. The curves $\rho = a \sin n\theta$, $\rho = a \cos n\theta$. (Give n values 1, 2, 3, 4.)

14. Show that the cardioid is described by a point on the circumference of a circle of radius a which rolls upon a fixed circle of the same radius.

91. Curvature. If a point moves along a plane curve as EF, Fig. 45, the direction of motion at any point is the direction of the tangent to the curve. The direction of the tangent continually changes, and a comparison of this change of direction with the distance traversed by the point leads us to the idea of curvature. Thus, as the point moves from P to Q, the tangent turns through the angle $\Delta\phi$, and we say the curvature of the arc PQ is large or small according as the angle $\Delta\phi$ is large or small.

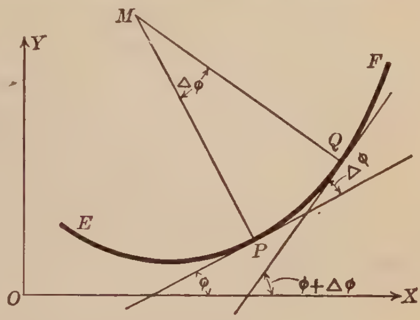

FIG. 45.

Denoting by Δs the length of the arc PQ, the ratio $\frac{\Delta\phi}{\Delta s}$ is defined as the **mean curvature** of the arc PQ. Provided the curvature is constant, this quotient is the curvature at all points between P and Q; but if the curvature is not constant, then the **curvature at** P is defined as the limit

$$\underset{\Delta s \doteq 0}{L} \frac{\Delta\phi}{\Delta s} = D_s\phi = \frac{d\phi}{ds}. \tag{1}$$

The curvature of the curve $y=f(x)$ at the point (x_1, y_1) may be expressed in terms of the derivatives $f'(x)$ and $f''(x)$. We have the fundamental relation

$$\tan\phi = f'(x), \quad \phi = \text{arc}\tan f'(x),$$

whence by differentiation

$$D_x\phi = \frac{D_x f'(x)}{1+[f'(x)]^2} = \frac{f''(x)}{1+[f'(x)]^2}.$$

Furthermore, we have

$$D_x s = \sqrt{1+[f'(x)]^2}.* \qquad \text{(See Art. 47)}$$

* A derivation of this relation is given in Art. 108. For the present, the student may make use of Eq. (1), Art. 47.

From these equations, we obtain by division

$$\frac{d\phi}{ds}=\frac{D_x\phi}{D_xs}=\frac{f''(x)}{\{1+[f'(x)]^2\}^{\frac{3}{2}}}. \tag{2}$$

At the point (x_1, y_1), therefore, the curvature is

$$\frac{f''(x_1)}{\{1+[f'(x_1)]^2\}^{\frac{3}{2}}}. \tag{3}$$

92. Radius of curvature. Center of curvature. Suppose that PQ, Fig. 45, is an arc of a circle. Let normals be drawn at P and Q intersecting at M; then $MP = MQ$ is the radius of the arc, and angle $PMQ = \Delta\phi$. Denoting the radius by r, we have

$$r\,\Delta\phi = \Delta s,$$

whence

$$\frac{\Delta\phi}{\Delta s}=\frac{1}{r}; \tag{1}$$

that is, the curvature of a circle is constant and is the reciprocal of the radius.

At any point of a curve $y = f(x)$, conceive a circle drawn tangent to the curve, and suppose it to have the same curvature as the curve at that point. This circle is called the **circle of curvature**; its radius, the **radius of curvature**; and its center, the **center of curvature** of the curve at the given point. Since the circle and curve have the same curvature at the point in question, the radius of curvature R must be the reciprocal of this curvature; that is,

$$R=\frac{ds}{d\phi}.$$

In terms of the derivatives involved, the formula for the radius of curvature at the point (x_1, y_1) becomes, therefore,

$$R=\frac{\{1+[f'(x_1)]^2\}^{\frac{3}{2}}}{f''(x_1)}. \tag{2}$$

By hypothesis, the curve and the circle of curvature at the given point have a common tangent; hence the radius of curvature has the direction of the normal to the curve.

It is customary in the case of single-valued functions to regard R positive or negative according as the curve is concave upward or concave downward. To establish this convention, it is necessary to take the positive value of the radical in the numerator. The sign of R then depends upon that of $f''(x)$, and the required result follows from Art. 87. In many cases it is the numerical value of R alone that is of importance.

As we have seen, $f''(x)=\dfrac{d^2y}{dx^2}$ changes sign at a point of inflexion; hence R also changes sign at such a point. This property might have been used as the definition of a point of inflexion. Thus a point of inflexion is a point at which the radius of curvature, and consequently the curvature itself, changes sign.

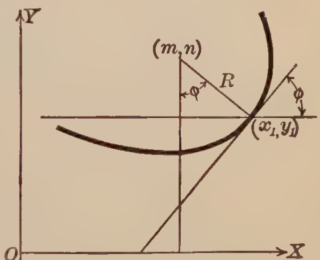

FIG. 46.

The coördinates of the center of curvature may be found as follows: If (m, n), Fig. 46, is the center of curvature corresponding to the point (x_1, y_1) of the curve, we have from the figure,

$$x_1 - m = R \sin\phi,\quad n - y_1 = R\cos\phi.$$

But
$$\tan\phi = f'(x_1),$$

whence
$$\cos\phi = \frac{1}{\sqrt{1+[f'(x_1)]^2}},\quad \sin\phi = \frac{f'(x_1)}{\sqrt{1+[f'(x_1)]^2}}.$$

Using these expressions for $\sin\phi$ and $\cos\phi$, and the expression for R given by (2), we obtain after reduction

$$\left.\begin{aligned} m &= x_1 - \frac{f'(x_1)\{1+[f'(x_1)]^2\}}{f''(x_1)};\\ n &= y_1 + \frac{1+[f'(x_1)]^2}{f''(x_1)}. \end{aligned}\right\} \qquad (3)$$

Ex. Find the radius of curvature of the equilateral hyperbola $xy = c^2$ at the point (x_1, y_1); also at the point (c, c).

We have
$$y = \frac{c^2}{x},\quad f'(x_1) = -\frac{c^2}{x_1^2},\quad f''(x_1) = \frac{2c^2}{x_1^3}.$$

Substituting these values in (2), we get

$$R = \frac{\left[1+\frac{c^4}{x_1^4}\right]^{\frac{3}{2}}}{\frac{2c^2}{x_1^3}} = \frac{(x_1^2+y_1^2)^{\frac{3}{2}}}{2c^2}.$$

At the point (c, c), therefore, $R = \frac{(2c^2)^{\frac{3}{2}}}{2c^2} = c\sqrt{2}.$

Also, at this point $f'(x) = -1$, and $f''(x) = \frac{2}{c}\cdot$ Substituting in (3), we find for the coördinates of the center of curvature,

$$m = n = 2c.$$

EXERCISES

Derive general expressions for the radius of curvature of each of the following curves:

1. $y^2 = 2ax$. **2.** $ay^3 = x^2 + b$. **3.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4. $y = \log(x + a)$. **5.** $y = \log \sec x$. **6.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

7. $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. **8.** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **9.** $y^3 = x^{\frac{3}{2}} - 5$.

10. $y^2 = \frac{x^3}{2a - x}\cdot$ **11.** $\sqrt{x} + \sqrt{y} = 2\sqrt{c}$.

For the following curves, find the radius of curvature and the coördinates of the center of curvature at the points indicated.

12. $y^2 = 10x$, at $x = 5$. **13.** $xy = 30$, at $(3, 10)$. **14.** $y = \cos x$, at $(0, 1)$.

15. Find the radius of curvature of the probability curve $y = ke^{ax^2}$ for $x = 0$.

16. Derive an expression for the curvature of the parabola $y = ax^2 + bx + c$, and show that the curvature is maximum at the vertex.

17. Find the point of maximum curvature of the exponential curve $y = e^x$.

93. Radius of curvature, parametric representation. If a curve is given by parametric equations, as

$$x = f(\theta), \quad y = \phi(\theta),$$

we may obtain an expression for the radius of curvature as follows:

We have the general formula

$$\frac{dy}{dx}=\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}, \tag{1}$$

from which, by the aid of Art. 31, we obtain

$$\frac{d^2y}{dx^2}=\frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2}-\dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^2}\frac{d\theta}{dx}$$

$$=\frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2}-\dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^3}. \tag{2}$$

Substituting these expressions in formula (2), Art 92, we obtain

$$R=\frac{\left[\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2}-\dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}. \tag{3}$$

Ex. Find the radius of curvature of the cycloid

$$x=a(\theta-\sin\theta),$$
$$y=a(1-\cos\theta).$$

By successive differentiations, we obtain

$$\frac{dx}{d\theta}=a(1-\cos\theta), \qquad \frac{dy}{d\theta}=a\sin\theta,$$

$$\frac{d^2x}{d\theta^2}=a\sin\theta, \qquad \frac{d^2y}{d\theta^2}=a\cos\theta.$$

Substituting these values in (3), we find

$$R=\frac{a^3 2^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{a^2(\cos\theta-1)}=-a\sqrt{8}\sqrt{1-\cos\theta}=-4a\sin\frac{\theta}{2}.$$

If a curve is given in polar coördinates, we have

$$x=\rho\cos\theta,\quad y=\rho\sin\theta.$$

By means of these relations (3) is reduced to the form

$$R = \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}. \tag{5}$$

Using the functional symbols for the derivatives, the radius of curvature of the curve $\rho = F(\theta)$ at the point (ρ_1, θ_1) is

$$R = \frac{\{\rho_1{}^2 + [F'(\theta_1)]^2\}^{\frac{3}{2}}}{\rho_1{}^2 + 2[F'(\theta_1)]^2 - \rho_1 F''(\theta_1)}. \tag{6}$$

EXERCISES

1. Find the radius of curvature of the ellipse from the equations

$$x = a\cos\theta, \qquad y = b\sin\theta.$$

2. Find the radius of curvature of the hypocycloid from the equations

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta.$$

Derive general expressions for the radii of curvature of the curves given by the following equations in polar coördinates.

3. $\rho = a\theta$.

4. $\rho = a(\sin\theta + \cos\theta)$.

5. $\rho = a\sin^3\frac{\theta}{3}$.

6. $\rho = e^{a\theta}$.

7. Find the maximum radius of curvature of the cardioid $\rho = 2a(1 - \cos\theta)$.

8. Find the minimum radius of curvature of the lemniscate $\rho^2 = a^2\cos 2\theta$.

9. From the answer to Ex. 1 show that the radius of curvature of the ellipse is greatest at the extremity of the minor axis and least at the extremity of the major axis.

94. Roulettes and involutes. Suppose a plane curve, as AB, Fig. 47, to roll without slipping on a fixed curve MN; for this purpose we may regard the curves as the boundaries of two disks. Points E, F on the rolling curve describe curves e, f on the fixed plane. Curves generated in this manner are called **roulettes.** Cycloids and trochoids are examples of roulettes in which the rolling curve is a circle. (See Art. 90.)

If the rolling curve is replaced by a straight line, Fig. 48, the roulettes are called **involutes** of the fixed curve; thus, curves e and f are involutes of the curve MN. Evidently a curve has an infinite number of involutes.

We may also consider the involute to be generated by a point on a flexible thread or cord wrapped around the fixed curve. As the cord is unwrapped, any point of it generates an involute.

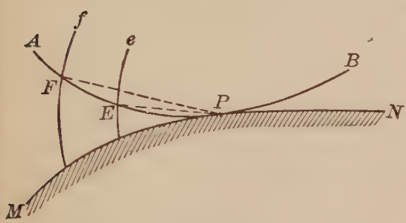

FIG. 47.

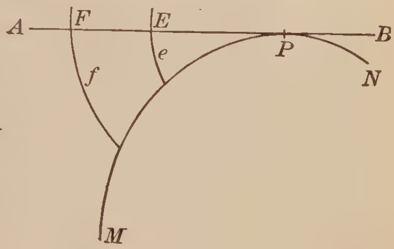

FIG. 48.

Certain properties of involutes are evident from the manner in which they are generated.

1. *Any normal to an involute is a tangent of the fixed curve.* This property may be seen from purely mechanical principles. Referring to Fig. 48, the point of contact P, considered as a point of the rolling line AB, has at the instant of contact no motion in the direction of the line, since by hypothesis there is no slipping. Furthermore, since the curves remain in contact, P can have no motion perpendicular to AB. It follows that the point P of AB is at rest, and the curve as a whole is rotating about P as a center. Points E and F are therefore moving in directions perpendicular to the lines PE and PF respectively. Now the point E moves in the direction of the tangent to the curve it describes; hence the line PE is the normal to the curve e at E, and likewise PF is normal to curve f at F. In the case of the involute, Fig. 48, the normals PE and PF coincide with the rolling line AB, which is tangent to MN; hence the normal of the involute is tangent to the curve.

1. *Two involutes of the same curve intercept a constant distance on their common normal.* This follows at once from the manner in which the involute is generated. Because of this property, involutes are sometimes called parallel curves.

95. Curvature of involutes. Let PQ, Fig. 49, be one position of the rolling straight line, P being the point of contact, and Q a

point on the involute. Let m, n denote the coördinates of the variable point P on the fixed curve MN, and x, y those of the point Q on the involute. From the geometry of the figure, we have at once

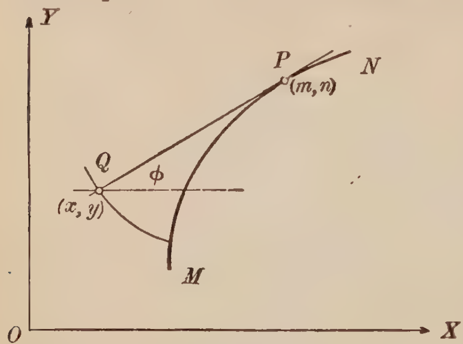

FIG. 49.

$$(m-x)\tan\phi = n-y. \quad (1)$$

Since QP is tangent to MN, we have

$$\tan\phi = \frac{dn}{dm};$$

but since QP is the normal to the involute at the point Q, we have also $\tan\phi = -\frac{dx}{dy}$. Substituting this value in (1) we obtain

$$(m-x)+(n-y)\frac{dy}{dx}=0. \quad (2)$$

Differentiating (2) with respect to n, we have

$$\frac{dm}{dn}-\frac{dx}{dn}+\left(1-\frac{dy}{dn}\right)\frac{dy}{dx}+(n-y)\frac{d\left(\frac{dy}{dx}\right)}{dn}=0.$$

But since $$\frac{dm}{dn}=-\frac{dy}{dx},$$

this equation reduces to

$$-\frac{dx}{dn}-\frac{dy}{dn}\frac{dy}{dx}+(n-y)\frac{d\left(\frac{dy}{dx}\right)}{dn}=0;$$

and multiplying through by $\frac{dn}{dx}$, we obtain finally,

$$-1-\left(\frac{dy}{dx}\right)^2+(n-y)\frac{d^2y}{dx^2}=0,$$

whence

$$n = y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad (3)$$

Substituting this value of n in (2), we have

$$m = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}. \tag{4}$$

Comparing (3) and (4) with (3) of Art. 92, it appears that the point P is the center of curvature and PQ is the radius of curvature at the point Q of the involute.

It is not true of roulettes in general that the point of contact of the rolling curves is the center of curvature of the roulette. Thus in Fig. 47, the center of curvature of e at the point E lies somewhere on EP or EP produced, but not at P. Only in the case of the involute does the center of curvature coincide with the center of rotation.

96. Evolute of a curve. From the preceding article it appears that a given curve is the locus of the centers of curvature of each of its involutes. When two curves C_1 and C_2 are so related that C_1 is an involute of C_2, we call C_2 the **evolute** of C_1. Curve C_2 may have other involutes than C_1, but C_1 has only the one evolute C_2.

Two properties of the evolute follow from the manner of describing the involute as a roulette, namely:

1. *Any normal to the curve C_1 is tangent to the evolute C_2.*

2. *The difference between two radii of curvature of a given curve is equal to the arc of the evolute between the points of contact of the radii with the evolute.*

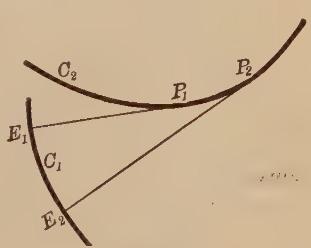

FIG. 50.

Thus in Fig. 50, we have

$$P_2E_2 - P_1E_1 = \text{arc } P_1P_2.$$

To obtain the equation of the evolute of a given curve, we combine the equation of the curve

$$y = f(x) \tag{1}$$

with the equations

$$n = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \tag{2}$$

$$m = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \tag{3}$$

which give the coördinates m, n of the center of curvature. If x, y, and the derivatives be eliminated between these equations, the result will be a relation between m and n, the variable coördinates of the evolute. Various special expedients may be used in the elimination.

Ex. 1. Find the evolute of the rectangular hyperbola $xy = \frac{a^2}{2}$.

We have $$\frac{dy}{dx} = -\frac{a^2}{2x^2}; \quad \frac{d^2y}{dx^2} = \frac{a^2}{x^3}.$$

Hence $$n = \frac{a^2}{2x} + \frac{4x^4 + a^4}{4a^2x} = \frac{4x^4 + 3a^4}{4a^2x},$$

$$m = x + \frac{4x^4 + a^4}{8x^3} = \frac{12x^4 + a^4}{8x^3}.$$

Therefore $$m + n = \frac{8x^6 + 12x^4a^2 + 6x^2a^4 + a^6}{8a^2x^3}$$

$$= a\left(\frac{a^2 + 2x^2}{2ax}\right)^3,$$

and $$m - n = a\left(\frac{a^2 - 2x^2}{2ax}\right)^3.$$

Finally $$(m + n)^{\frac{2}{3}} - (m - n)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

which is the desired equation of the evolute.

EXERCISES

Find the equations of the evolutes of the following curves.

1. The circle $x^2 + y^2 = a^2$.
2. The parabola $y^2 = 4px$.
3. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4. Find the evolute of the ellipse using the parametric equations $x = a\cos\theta$, $y = b\sin\theta$.

5. Find the evolute of the curve given by the parametric equations $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

MISCELLANEOUS EXERCISES

Find the radius of curvature and coördinates of the center of curvature of the following curves at the points indicated.

1. $y^2 = 8x + 1$, at $(6, 7)$. **2.** $y = 3x^3 - 8x + 4$, at $(-2, -4)$.

3. $y = \cos x$, at $(0, 1)$. **4.** $y = xe^x$, at $(0, 0)$.

5. Show that the evolute of an arch of the cycloid consists of the halves of an equal cycloid.

6. Show that the radius of curvature at any point of a cycloid is double the length of the normal at the same point.

7. If the equation of a curve can be written in the form

$$y = \pm(ax + b) + \phi(x),$$

where $\phi(x)$ is a rational fraction the denominator of which is higher degree than the numerator, show that the lines $y = \pm(ax + b)$ are asymptotes to the curve. In this way determine the asymptotes to the curve

$$x^3 - xy^2 + ay^2 = 0.$$

8. Show that the ordinates of the curve $x^3 - xy + 1 = 0$ and of the parabola $y = x^2$ approach equality as x increases. For this reason, the parabola is said to be a *curvilinear asymptote* to the curve.

9. Trace the curve $y + xy - x^3 = 0$. Find its rectilinear and curvilinear asymptotes.

10. Show that at the point $(0, 0)$ the curve $y = \dfrac{x}{1 + e^{\frac{1}{x}}}$ has two terminating branches with different tangents.

Examine for singular points the following curves.

11. $x(x - a)^2 + y^2(x - 2a) = 0$.

12. $(x^2 + y^2)^2 = 4x^2 + y^2$.

13. $x^4 - 3xy + 6y^2 - y^4 = 0$.

14. By the method of limiting intercepts, find the asymptotes to the curves (a) $y^3 = 4x^2 + x^3$; (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

15. Find the radius of curvature of the three-cusped hypocycloid from the parametric equations

$$x = a(2\cos\theta + \cos 2\theta),\quad y = a(2\sin\theta - \sin 2\theta).$$

16. The tractrix is a curve having the property that the length of the tangent is a constant a. Find the equation of this curve.

17. Show that a curve whose equation is given in polar coördinates has an asymptote when the subtangent $\rho^2 \dfrac{d\theta}{d\rho}$ is finite for $\rho = \infty$.

18. Using the result of Ex. 17, find the asymptotes of the following curves:

$$(a)\ \rho = a \tan\theta; \quad (b)\ \rho = \frac{2\pi}{\sqrt{\theta}}; \quad (c)\ \rho = a \sec 2\theta.$$

19. Prove that the evolute of the logarithmic spiral $\rho = ae^{k\theta}$ is a similar logarithmic spiral.

20. Find the equation of the evolute of the cissoid

$$y^2 = \frac{x^3}{2a - x}.$$

21. A circle of radius b rolls on a fixed circle of radius a, $a > b$, and a point on the rolling circle generates an *epicycloid*. Show (a) that the equations of the curve are

$$x = (a + b)\cos\phi - b\cos\frac{a+b}{b}\phi,$$

$$y = (a + b)\sin\phi - b\sin\frac{a+b}{b}\phi;$$

(b) that the radius of curvature is $\dfrac{4b(a+b)}{a+2b}\sin\dfrac{a\phi}{2b}$.

22. Discuss the family of curves obtained by giving p different constant values in the equation $c = \alpha + \beta T + p\left(1 + \dfrac{a}{2}p\right)\dfrac{C}{T^{n+1}}$, which gives the relation between specific heat c and temperature T of superheated steam.

23. Trace the curve $y^2 = \dfrac{x^3}{x - a}$. Examine for maxima and minima, asymptotes, points of inflexion, and cusps.

24. Van der Waals' equation $p = \dfrac{RT}{v - b} - \dfrac{a}{v^2}$ gives the relation between pressure, volume, and absolute temperature of certain substances. Giving T different constant values, a family of curves called *isothermals* are obtained. For carbon dioxide $R = 0.00369$, $a = 0.00874$, $b = 0.0023$. Trace the isothermals for $T = 250, 300, 350$.

CHAPTER X

DEFINITE INTEGRALS

97. Definition of a definite integral. Suppose we have given a function $f(x)$ which is continuous and single-valued within a given interval (a, b), and suppose further that

$$\int f(x)\,dx = D_x^{-1}f(x) = \phi(x) + C. \tag{1}$$

The difference of the values of the function $\phi(x) + C$ for any two values of the independent variable is called a **definite integral.** The values of the independent variable substituted are called the **limits of integration.** We denote a definite integral symbolically by writing the two limits of integration at the extremities of the sign of integration. Thus, if $x = a$ and $x = b$ are the limits of integration in the definite integral formed from (1), we indicate that fact by writing this definite integral as follows:

$$\int_a^b f(x)\,dx = \left[D_x^{-1}f(x)\right]_a^b = \phi(b) - \phi(a). \tag{2}$$

This symbol is read: "The definite integral of $f(x)$ between the limits $x = a$ and $x = b$," or "from a to b." It is to be noted that in the definite integral the constant of integration disappears. This result follows from the definition; for, we have

$$\left[\phi(x) + C\right]_a^b = \left[\phi(b) + C\right] - \left[\phi(a) + C\right] = \phi(b) - \phi(a). \tag{3}$$

To distinguish definite integrals from those previously discussed, we call the latter **indefinite integrals.** We may not only pass from the indefinite to the definite integral by the process already indicated, but conversely we may pass from the definite to the indefinite form of the integral by assuming the upper limit of integration as variable. Thus, we have

$$\int_a^x f(x)\,dx = \phi(x) - \phi(a),$$

where $\phi(a)$ may be taken as the constant of integration.

EXERCISES

Verify the following:

1. $\int_1^5 4x^3\,dx = 624.$

2. $\int_1^2 (3x - x^3)\,dx = \frac{3}{4}.$

3. $\int_0^\beta \sin\theta\,d\theta = 1 - \cos\beta.$

4. $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}.$

5. $\int_{v_1}^{v_2} \frac{dv}{v} = \log\frac{v_2}{v_1}.$

6. $\int_1^4 \frac{dy}{y} = \log 4.$

7. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan\phi\,d\phi = 0.$

8. $\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,d\theta = \frac{1}{2}.$

Evaluate the following definite integrals:

9. $\int_1^3 \frac{x\,dx}{3+x^2}.$

10. $\int_2^4 x^{-\frac{3}{2}}\,dx.$

11. $\int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta.$

12. $\int_0^{\frac{\pi}{2}} \sin x\,dx.$

13. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta.$

14. $\int_0^r \left(\frac{r^2}{a^2-x^2}\right)^{\frac{1}{2}} dx,\ r < a.$

15. $\int_1^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})\,x^{-\frac{1}{3}}\,dx.$

16. $\int_0^\pi \sin\theta\,d\theta.$

17. $\int_0^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{1+\sin^2\theta}.$

18. $\int_0^1 \sqrt{1+9x^4}\cdot x^3\,dx.$

19. Show from known formulas of physics that the definite integral $\int_{t_1}^{t_2} gt\,dt$ gives the space traversed by a falling body in the time interval $t_2 - t_1$.

20. Plot the curve $v = f(t) = gt$ and show from geometry that the area between this curve, the t-axis, and the ordinates at t_1 and t_2 is numerically equal to $\int_{t_1}^{t_2} gt\,dt$.

21. Evaluate $\int_1^4 (3x^2 - 4)\,x\,dx$ and $\int_4^1 (3x^2-4)\,x\,dx$. What is the relation between the two integrals? State a general law covering the interchange of the limits of integration.

22. Show that $\int_2^5 (x^2 - 3x)\,dx + \int_5^8 (x^2 - 3x)\,dx = \int_2^8 (x^2 - 3x)\,dx.$ What general law does this result suggest?

23. Show that the general theorems for indefinite integrals (Art. 67) apply also to definite integrals.

98. Elementary properties of definite integrals. The general properties of integrals already developed apply equally well to definite integrals. Thus, the definite integral of a constant times a function is equal to that constant times the definite integral of the function; the sum of a finite number of definite integrals having the same limits, is equal to the definite integral of the sum of the functions, etc. In addition to these, there are certain properties which apply to definite integrals alone. Among the more elementary of these properties are the following:

THEOREM I. *Interchanging the limits of integration changes the sign of the integral;* that is,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

From the definition of a definite integral, we have

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a), \tag{1}$$

and

$$\int_b^a f(x)\,dx = \phi(a) - \phi(b). \tag{2}$$

Hence,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{3}$$

THEOREM II. *If c lies between a and b, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

For, we have

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a), \tag{4}$$

$$\int_a^c f(x)\,dx = \phi(c) - \phi(a), \tag{5}$$

$$\int_c^b f(x)\,dx = \phi(b) - \phi(c). \tag{6}$$

Adding (5) and (6), we obtain

$$\begin{aligned}\int_a^c f(x)\,dx + \int_c^b f(x)\,dx &= \phi(c) - \phi(a) + \phi(b) - \phi(c)\\ &= \phi(b) - \phi(a)\\ &= \int_a^b f(x)\,dx, \end{aligned} \tag{7}$$

which establishes the theorem. Evidently the theorem may be extended to include any finite number of values between a and b.

EXERCISES

1. Evaluate the integral $\int (3x^2 + 5)\,dx$, using successively the limits of integration 1 to 3, 3 to 4, 4 to 6, and 1 to 6, and verify Theorem II by the result.

2. By evaluating the integrals, show that

$$\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin\theta\,d\theta = \int_0^{\pi} \sin\theta\,d\theta.$$

3. Show that $\int_a^b \phi(u)\,du = \int_a^b \phi(x)\,dx$. State this result in the form of a theorem.

99. Change of limits. In the process of finding an integral, it is sometimes convenient to change the variable (see Art. 71). In this case, we may by properly changing the limits of integration obtain the definite integral without a second substitution. Suppose we have the integral $\int_a^b f(x)\,dx$ and make the substitution $z = F(x)$. For $x = a$, we have $z = F(a)$, and for $x = b$, $z = F(b)$; hence the substitution of $F(a)$ and $F(b)$ as limits of integration in the transformed integral must lead to the same result as the substitution of a and b in the original integral.

Ex. Find $\int_0^{\frac{1}{2}} \frac{\text{arc}\sin x\,dx}{\sqrt{1-x^2}}$, assuming $z = \text{arc}\sin x$.

We have $dz = \frac{dx}{\sqrt{1-x^2}}$, whence $\frac{\text{arc}\sin x\,dx}{\sqrt{1-x^2}} = z\,dz$.

For $x = 0$, $z = \text{arc}\sin 0 = 0$, the lower limit, and for $x = \frac{1}{2}$, $z = \text{arc}\sin\frac{1}{2} = \frac{\pi}{6}$, the upper limit.

Hence
$$\int_0^{\frac{1}{2}} \frac{\text{arc}\sin x\,dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{6}} z\,dz = \frac{\pi^2}{72}.$$

EXERCISES

1. $\int_0^2 \frac{x\,dx}{1+x^4}$. Let $z = x^2$.

2. $\int_0^{\frac{a}{2}} \frac{dx}{\sqrt{a^2-x^2}}$. Let $z = \frac{x}{a}$.

3. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx.$ Let $z = \sin x$.

4. $\int_0^2 e^{x^2} x \, dx.$ Let $z = x^2$.

5. $\int_0^1 \frac{dx}{e^x + e^{-x}}.$ Let $z = e^x$.

6. $\int_0^{\frac{\pi}{4}} \frac{\sin\theta \, d\theta}{\cos^3\theta}.$ Let $z = \tan\theta$.

7. $\int_1^2 \frac{(\log^2 x)\, dx}{x}.$ Let $z = \log x$.

8. $\int_0^a \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}.$ Let $z = \text{arc} \tan \frac{x}{a}$.

100. Definite integral as the limit of a sum. In Art. 7 it was shown that the area underneath a curve might be obtained as a limit of the sum of certain rectangular elements. As pointed out in that connection, the method there employed is impracticable and except in the most elementary cases, perhaps, impossible. We shall now show the relation of the summation process to the definite integral; in fact, we shall show that under certain restrictions as to the character of the function, the limit of the sum of an indefinitely large number of indefinitely small elements may always be replaced by a definite integral taken between the proper limits.

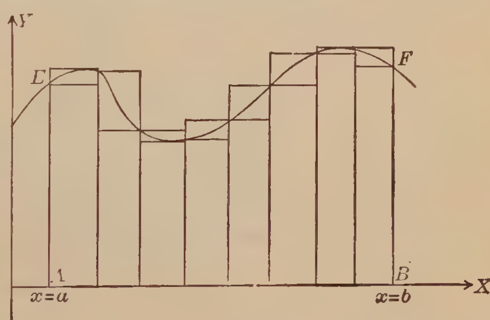

FIG. 51.

For this purpose, let $f(x)$ be a single-valued, continuous function in the interval $a \leqq x \leqq b$, and suppose it be represented by some curve as EF, Fig. 51. Let the interval (a, b) be divided into subintervals by the insertion in any manner whatever on the X-axis of the points $x_1, x_2, \cdots, x_{n-1}$ between A and B. Let each subinterval be multiplied by the value of $f(x)$ at some point within it.

Denoting these values of $f(x)$ by $f_1(x), f_2(x), \cdots, f_n(x)$ respectively, we have then the sum

$$\sum_1^n f_i(x)\Delta x_i \equiv (x_1 - a)\, f_1(x) + (x_2 - x_1)\, f_2(x) + \cdots + (b - x_{n-1}) f_n(x), \tag{1}$$

where Δx_i denotes any one of the subintervals $(x_i - x_{i-1})$. In each subinterval Δx_i, let α_i, β_i denote the smallest and largest numerical values, respectively, of $f_i(x)$. With Δx_i as a base let two rectangles be constructed having α_i and β_i respectively as their altitudes. Repeating this construction for each subinterval, we have the relation

$$\sum_1^n \alpha_i \Delta x_i \leqq \sum_1^n f_i(x)\Delta x_i \leqq \sum_1^n \beta_i \Delta x_i. \tag{2}$$

The first and the last of these sums have limits as n increases indefinitely since both are monotone functions and limited in magnitude by the conditions placed upon $f(x)$. Moreover these limits are equal, say equal to A, and that independently of the manner of subdividing the given interval (a, b).* Since $\Delta x \doteq 0$ as n becomes infinite, we may now write (see Art. 13)

$$\underset{\Delta x \doteq 0}{L} \sum_a^b f(x)\Delta x = A. \tag{3}$$

The existence of the limit in (3) being established, we may find the value of the limit by taking the subintervals and the corresponding values of the function in any particular manner we please. It is convenient to make the values of Δx equal and to take the value of $f(x)$ at the beginning of each interval. We have then for all values of n,

$$n\,\Delta x = b - a, \tag{4}$$

and the original points of division become

$$a + \Delta x, \quad a + 2\,\Delta x, \cdots, \quad b - \Delta x.$$

In defining the definite integral, $\phi(x)$ was taken as a function having $f(x)$ as its derivative. From the law of the mean, we have therefore

$$\phi(x + \Delta x) - \phi(x) = \Delta x \cdot f(x + \theta \cdot \Delta x), \quad 0 < \theta < 1. \tag{5}$$

* See Picard's *Traité d'Analyse*, Vol. 1, p. 4; or Veblen and Lennes' *Infinitesimal Analysis*, p. 150 et seq.

By hypothesis α and β are respectively the minimum and maximum values of $f(x)$ in a subinterval; hence we may write

$$\alpha \leqq f(x+\theta \cdot \Delta x) \leqq \beta.$$

By aid of (5) this inequality becomes

$$\alpha \cdot \Delta x \leqq \phi(x+\Delta x) - \phi(x) \leqq \beta \cdot \Delta x. \tag{6}$$

We may now apply (6) to the successive subintervals by substituting for x the values

$$a, \quad a+\Delta x, \quad a+2\,\Delta x, \cdots, \quad b-\Delta x.$$

The resulting inequalities are:

$$\alpha_1\,\Delta x \leqq \phi(a+\Delta x) - \phi(a) \leqq \beta_1\,\Delta x,$$

$$\alpha_2\,\Delta x \leqq \phi(a+2\,\Delta x) - \phi(a+\Delta x) \leqq \beta_2\,\Delta x,$$

$$\cdots\cdots\cdots\cdots$$

$$\alpha_n\,\Delta x \leqq \phi(b) - \phi(b-\Delta x) \leqq \beta_n\,\Delta x.$$

Adding these results, we have

$$\sum_a^b \alpha_i\,\Delta x \leqq \phi(b) - \phi(a) \leqq \sum_a^b \beta_i\,\Delta x. \tag{7}$$

Since we have

$$\underset{\Delta x \doteq 0}{L} \sum_a^b \alpha\,\Delta x = \underset{\Delta x \doteq 0}{L} \sum_a^b \beta\,\Delta x = A, \tag{8}$$

it follows that

$$\phi(b) - \phi(a) = A, \tag{9}$$

and we may therefore write

$$\underset{\Delta x \doteq 0}{L} \sum_a^b f(x)\,\Delta x = \phi(b) - \phi(a) = \int_a^b f(x)\,dx. \tag{10}$$

We may henceforth replace at any time the summation process by the definite integral, and regard the two symbols, "$\underset{\Delta x \doteq 0}{L} \sum_a^b$" and "$\int_a^b$" as interchangeable whenever the integrand is a continuous function.

From this discussion it also follows that the definite integral

$\int_a^b f(x)\,dx$ is represented graphically by the area bounded by the curve $y = f(x)$, the X-axis, and the two ordinates $x = a$, $x = b$.

101. Importance of the summation process. The result just derived and expressed by equation (10) of the preceding article is of the highest importance, for it enables us to replace a difficult and tedious process of direct summation by a process that is in most cases simple and easily carried out. As was shown in the example of Art. 7, the determination of the area under a curve involves the summation of an indefinitely large number of indefinitely small terms, that is, it requires the limit $\underset{\Delta x \doteq 0}{L} \sum_a^b f(x)\,\Delta x$. But according to (10) the limit is given by the definite integral $\int_a^b f(x)\,dx$. Hence to find the area we have only to find the anti-derivative $\phi(x)$ of the given function $f(x)$, substitute the limits a and b, and take the difference $\phi(b) - \phi(a)$. For example, to find the area under the curve $y = x^2$ from the origin to the ordinate $x = 3$ (see Art. 7), we have

$$A = \int_0^3 f(x)\,dx = \int_0^3 x^2\,dx = \tfrac{1}{3}\,x^3\Big]_0^3 = 9.$$

While the discussion leading to equation (10) was accompanied by a geometrical illustration, and the summation was directed toward the determination of an area, the course of reasoning depends in no way upon geometrical considerations. The method of the definite integral is applied with equal facility to the determination of magnitudes of all kinds — volumes, masses, fluid pressures, heat, work, etc. In the following chapters there will be given examples of the use of the summation process in finding the lengths of curves, the areas of the surfaces, the volumes of solids, etc. In mechanics, determinations of centers of gravity and moments of inertia likewise involve the summation principle. The work done by a variable force is found by the summation of terms of the type $F\Delta s$, where F denotes force and s displacement. The impulse of a variable force is the summation of terms of the type $F\Delta t$. The space over which a moving point travels is found by the summation of terms of the type $v\,\Delta t$, where v denotes the

velocity of the point. If the specific heat c of a substance varies with the temperature τ, the heat that must be imparted to the substance to produce a given rise of temperature is determined by the summation $\Sigma c\,\Delta\tau$. Other applications will occur to the student.

It should be noted that the summation of an infinite number of terms is always necessary when one of the factors entering into the problem varies continuously. As an illustration, take the problem of finding the mass of a body. If V denotes the volume of the body and γ its density, the product γV gives the mass, provided the density is constant throughout. The density, however, may differ for different parts of the body, as, for example, when the body is composed of different liquids which arrange themselves in layers or strata. If $V_1, V_2, V_3, \cdots, V_n$ denote the volumes of the separate parts, and $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_n$ the corresponding densities, the total mass is evidently the sum

$$\gamma_1 V_1 + \gamma_2 V_2 + \gamma_3 V_3 + \cdots + \gamma_n V_n.$$

In this case, the number of parts being finite, we need only simple addition. However, the density may vary continuously throughout the body, as in the case of the atmosphere. Here we must have recourse to the summation of an infinite number of indefinitely small terms. We divide the total volume V into n parts each equal to ΔV and multiply each element ΔV by the density at that part of the body. We thus get n terms of the type $\gamma\,\Delta V$. If n is finite, the sum of these n terms is not the exact value of the mass because the density varies in the element of volume ΔV. But as n is taken larger and ΔV correspondingly smaller, the sum of the n terms approaches more nearly the mass. Hence, to get the exact result, we must increase n indefinitely, thus making ΔV correspondingly small, and effect the summation of the infinitely large number of infinitesimal terms. That is, we must find

$$\underset{\Delta V \doteq 0}{L} \sum \gamma\,\Delta V.$$

As previously shown, the summation is effected most easily by means of the definite integral. The elements to be summed being of the type $f(x)\,\Delta x$, we find the anti-derivative $\phi(x)$ of the function $f(x)$, substitute the limits of integration, say a and b, and take

the difference $\phi(b) - \phi(a)$. The problem of effecting the summation reduces, therefore, to a problem in integration.

EX. A vertical wall (as a dam) having a height h and breadth b, Fig. 52, is subjected to water pressure, the intensity of which varies as the depth below the liquid surface. Required the total pressure on the wall.

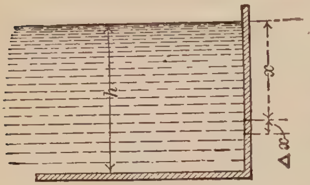

FIG. 52.

According to the law of liquid pressure, the intensity at the depth x is kx, where k is constant. Let the wall be divided into elements of width Δx and length b; then if the area of the strip $b\,\Delta x$ is multiplied by the intensity of pressure kx *at the top* of the strip, the product $b\,\Delta x \cdot kx = kb\,x\,\Delta x$ gives *approximately* the pressure on the element of area. The sum of a finite number of terms of the type $kb\,x\,\Delta x$ would give a total pressure somewhat smaller than the actual value; but the limit $\underset{\Delta x \doteq 0}{L} \sum_0^h kbx\,\Delta x$ evidently gives the exact result. This limit is the definite integral $\int_0^h kbx\,dx = kb\int_0^h x\,dx = \frac{1}{2}kbh^2$. Hence the total pressure on the wall is $\frac{1}{2}kbh^2$.

102. Geometrical representation of a definite integral. It was pointed out in Art. 100 that a definite integral $\int_a^b f(x)\,dx$ is represented by the area bounded by the curve $y = f(x)$, the X-axis, and the ordinates corresponding to $x = a$, $x = b$. Whatever magnitude the definite integral is used to denote, volume, mass, fluid pressure, work, or moment, this area is the graphical representation of it; that is, the number of units of area is the same as the number of units of the magnitude denoted by the integral. In fact, one way of evaluating a definite integral is to measure by some mechanical means the area that represents it. (See Art. 119.)

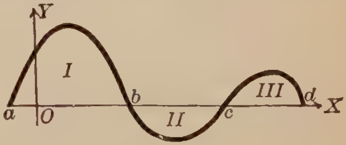

FIG. 53.

If $f(x)$ becomes negative for certain values of x, the graph of $f(x)$ will lie below the X-axis, as from b to c, Fig. 53. In this case the integral $\int_a^d f(x)\,dx$ is represented by the algebraic sum of the areas I, II, and III, area II being taken as negative. If

the numerical rather than the algebraic sum is desired, we must take the sum of the three integrals $\int_a^b f(x)\,dx$, $\int_b^c f(x)\,dx$, and $\int_c^d f(x)\,dx$ without reference to sign.

Ex. Show that the area which represents the total liquid pressure in the example of the preceding article is a triangle whose base is h and whose altitude is kbh. (Note that $f(x) = kbx$.)

EXERCISES

1. Give a geometric interpretation of Theorem II, Art. 98.

2. Give a geometric proof of the following theorem: If M and N are respectively the greatest and least values of the continuous function $f(x)$ within the interval (a, b), then, provided $b > a$,

$$N(b-a) < \int_a^b f(x)\,dx < M(b-a).$$

3. Give a geometric proof of the following theorem: If $b > a$, and $\phi(x)$, $f(x)$, and $\psi(x)$ are three functions such that for any value of x within the interval (a, b), $\phi(x) < f(x) < \psi(x)$, then

$$\int_a^b \phi(x)\,dx < \int_a^b f(x)\,dx < \int_a^b \psi(x)\,dx.$$

4. Using the theorem of Ex. 3, show that if $0 < n < 2$, $\int_0^1 \frac{dx}{1+x^n}$ lies between 0.5 and 0.7854.

5. Give a geometric proof of the theorem

$$\int_0^a F(x)\,dx = \int_0^a F(a-x)\,dx.$$

6. Show geometrically that $\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta.$

7. Show from geometric considerations that

$$\int_0^{2\pi} \sin\theta\,d\theta = 0.$$

8. Show the area that represents the definite integral $\int_{t_1}^{t_2} gt\,dt$, which applies to falling bodies.

103. Definite integrals of discontinuous functions: Infinite limits of integration. So far we have discussed definite integrals of continuous functions with finite limits of integration. We shall now consider cases where one or both of these conditions do not hold.

Suppose that $f(x)$ has a point of discontinuity at $x=c$, Fig. 54. The extent of this discontinuity may be finite or infinite. In such a case the integrals $\int_a^b f(x)\,dx$, $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ have no meaning according to the definition given for a definite integral. However, the definite integral does have a meaning for the intervals $(a,\ c-\epsilon)$, $(c+\epsilon,\ b)$; for within these intervals the function is continuous. We may, therefore, define the value of the definite integral for the intervals (a, c) and (c, b) as the limits

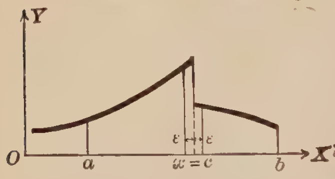

FIG. 54.

$$\underset{\epsilon \doteq 0}{L} \int_a^{c-\epsilon} f(x)\,dx, \qquad \underset{\epsilon \doteq 0}{L} \int_{c+\epsilon}^{b} f(x)\,dx,$$

provided these limits exist. The definite integral for the interval (a, b) may now be defined as the sum of these two limits. We may not always, however, obtain the proper value of the last integral by evaluating directly the integral $\int_a^b f(x)\,dx$.

Ex. 1. Find the area between the curve $y=\dfrac{1}{(x-2)^2}$, the X-axis, and the ordinates for which $x=0$, $x=4$.

This function is discontinuous for $x=2$, as is shown in Fig. 55. The definite integral for the given interval is found by taking the sum of the limits

$$\underset{\epsilon \doteq 0}{L} \int_0^{2-\epsilon} \frac{dx}{(x-2)^2} + \underset{\epsilon \doteq 0}{L} \int_{2+\epsilon}^{4} \frac{dx}{(x-2)^2}$$

$$= \underset{\epsilon \doteq 0}{L} \left.\frac{-1}{x-2}\right]_0^{2-\epsilon} + \underset{\epsilon \doteq 0}{L} \left.\frac{-1}{x-2}\right]_{2+\epsilon}^{4}$$

$$= \underset{\epsilon \doteq 0}{L} \frac{1}{\epsilon} - \frac{1}{2} - \frac{1}{2} + \underset{\epsilon \doteq 0}{L} \frac{1}{\epsilon}$$

$$= \underset{\epsilon \doteq 0}{L} \frac{2}{\epsilon} - 1 = \infty.$$

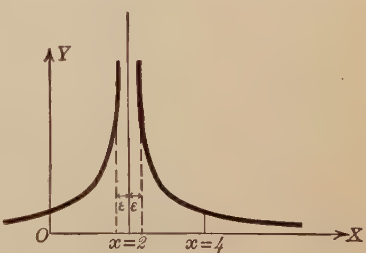

FIG. 55.

Hence the area in question is infinite.

Let us now take the integral directly between the limits 0 and 4. We obtain for the area

$$\int_0^4 f(x)\,dx = \int_0^4 \frac{dx}{(x-2)^2} = \left.\frac{-1}{x-2}\right]_0^4 = -1,$$

an incorrect result. In this case we say therefore that the integral $\int_0^4 f(x)\,dx$ has no meaning.

We must also consider the special case in which one of the limits of integration is infinite. This again is a limiting case of the ordinary definite integral, and we define the definite integral $\int_a^\infty f(x)\,dx$ as the limit $\underset{x=\infty}{L} \int_a^x f(x)\,dx$. Hence, provided this limit exists, the integral $\int_a^\infty f(x)\,dx$ has a meaning. The following example illustrates this case:

Ex. 2. Find the area between the X-axis and the witch of Agnesi, whose equation is

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

See Fig. 39, Art. 90. The curve is symmetrical with respect to the Y-axis and approaches the X-axis as x becomes infinite. Hence the area is given by

$$2\int_0^\infty y\,dx = \underset{x=\infty}{L}\, 16a^3 \int_0^x \frac{dx}{x^2+4a^2} = 16a^3 \underset{x=\infty}{L} \left[\frac{1}{2a}\arctan\frac{x}{2a}\right]_0^x$$

$$= 8a^2 \underset{x=\infty}{L} \arctan\frac{x}{2a} = 4\pi a^2.$$

EXERCISES

1. Evaluate $\int_0^\infty \frac{dx}{a^2+x^2}$.

2. Evaluate $\int_0^\infty e^{-ax}\,dx$.

3. Evaluate $\int_0^1 \frac{dx}{(1-x)^{\frac{1}{2}}}$.

4. Evaluate $\int_{-\infty}^{-1} \frac{dx}{x\sqrt{x^2-1}}$.

5. Of the following definite integrals, which have finite values?

$$(a)\ \int_0^1 \frac{dx}{x^2};\quad (b)\ \int_0^1 \frac{dx}{x};\quad (c)\ \int_a^\infty \frac{dx}{x^3};\quad (d)\ \int_0^1 \frac{dx}{1-x^2}.$$

6. Show that if $n < 1$, the area under the curve $x^n y = C$ from the origin to $x = a$ is finite, while for $n > 1$ the area under the curve from $x = a$ tc $x = \infty$ is finite.

7. With the data of Ex. 6 investigate the area when $n = 1$.

8. Find the area under the curve

$$y(x^2-1)^{\frac{2}{3}} = x$$

between $x = 0$ and $x = 4$. Draw the curve.

MISCELLANEOUS EXERCISES

Evaluate the following definite integrals:

1. $\int_0^{s_1} \sqrt{2\,gs}\,ds.$ **2.** $\int_0^a \left(x\sqrt{a^2-x^2}+\frac{a^2-x^2}{2}\right)dx.$

3. $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^8\theta\,d\theta.$ (Put $\cos^2\theta = 1-\sin^2\theta$.)

4. Show that $\dfrac{\int_0^a x(a-x)\,dx}{\int_0^a (a-x)\,dx} = \frac{1}{3}a.$

5. Evaluate (*a*) $\dfrac{\int_\alpha^\beta \sin\theta\cos\theta\,d\theta}{\int_\alpha^\beta \sin\theta\,d\theta}$; (*b*) $\dfrac{\int_0^{\frac{\pi}{2}} a\cos^2\theta\sin\theta\,d\theta}{\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta}.$

6. Find by the summation method of Art. 7 the area between the curve $y = e^x$, the X-axis, and the ordinates $x = 1$, $x = 4$.

7. Change the limits of integration of the following integrals when the variable is changed as indicated.

(*a*) $\int_4^{20}(\;)\,dx,\; x = z^2+4;$ (*b*) $\int_0^1(\;)\,dx,\; x = \sin\theta-\cos\theta;$

(*c*) $\int_0^{\log 10}(\;)\,dx,\; e^x = z^2+1;$ (*d*) $\int_0^a(\;)\,dx,\; x = a\sin\theta.$

8. Show from geometrical considerations that

(*a*) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx;$ (*b*) $\int_a^b f(x)\,dx = \int_0^{b-a} f(x+a)\,dx.$

Evaluate the following:

9. $\int_0^2 \frac{dx}{\sqrt{2-x}}.$ **10.** $\int_0^a \frac{dx}{\sqrt{a^2-x^2}}.$ **11.** $\int_0^4 \frac{dx}{(x-2)^2}.$

12. Prove and interpret geometrically the following theorem:

(*a*) If $f(-x) = -f(x)$, $\int_{-a}^a f(x)\,dx = 0.$

(*b*) If $f(-x) = f(x)$, $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$

13. Apply the theorems of Ex. 12 to the definite integrals

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta;\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,d\theta.$$

14. If $n > 2$, show that $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}}$ lies between 0.5 and 0.52.

15. If $f(x) = \sqrt{8x + 10}$, show that $\int_5^8 f(x)\,dx$ must lie between $3\sqrt{50}$ and $3\sqrt{74}$.

16. Find the limits between which the integral $\int_3^5 f(x)\,dx$ must lie where $f(x) = \log x$.

17. Prove the following theorem geometrically:

$$\int_a^b \phi(x)\,dx = (b-a)\phi[a + \theta(b-a)],$$

where $0 < \theta < 1$.

CHAPTER XI

APPLICATIONS OF INTEGRATION TO GEOMETRY AND MECHANICS

104. Plane areas, rectangular coördinates. If the equation of a curve in rectangular coördinates is $y=f(x)$, then, as seen in Art. 100, the area A between the curve, the X-axis, and the ordinates $x=a$ and $x=b$, respectively, is given by the formula

$$A=\int_a^b f(x)\,dx=\int_a^b y\,dx. \tag{1}$$

Ex. **1.** Find the area included between the parabola $y^2=8x$, the X-axis, the origin, and the ordinate $x=18$.

Since $y^2=8x$, $y=\sqrt{8x}$, and

$$A=\int_0^{18} y\,dx=\sqrt{8}\int_0^{18} x^{\frac{1}{2}}\,dx=\sqrt{8}\left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^{18}=144.$$

Ex. **2.** Find the area under one arch of the curve $y=\cos x$.

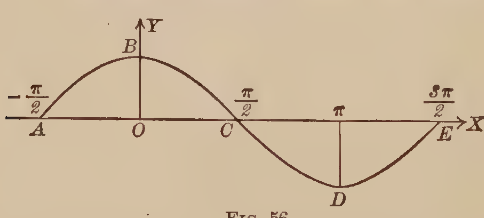

FIG. 56.

From the graph of the curve, Fig. 56, it is seen that $y=0$ for $x=-\frac{\pi}{2}$ and for $x=\frac{\pi}{2}$. Hence, we have

$$\text{area } ABC=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=\sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2.$$

Likewise, $$\text{area } CDE=\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos x\,dx=\sin x\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=-2.$$

Also, area $ABCDE = \sin x\Big]_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0$, and area $BCD = \sin x\Big]_0^{\pi} = 0.$

The last two results are consistent when the signs of the areas are taken into account. Thus areas ABC and CDE are equal in magnitude but opposite in sign. The numerical magnitude of the area $ABCDE$ is 4.

It will be observed that the area ABC is symmetrical with respect to the axis OY; consequently

$$\text{area } ABC = 2 \times \text{area } OBC = 2\int_0^{\frac{\pi}{2}} \cos x\,dx = 2.$$

Likewise, $$\text{area } CDE = 2\int_{\frac{\pi}{2}}^{\pi} \cos x\,dx = -2.$$

In general, it is advisable to make use of conditions of symmetry as far as possible, and take the narrowest interval of integration that the problem permits.

Frequently it is desirable to find the area between the curve, the Y-axis, and the abscissas corresponding to $y = \alpha$ and $y = \beta$, respectively. The formula for this area is

$$A = \int_\alpha^\beta x\,dy. \qquad (2)$$

The derivation of form (2) follows precisely that of form (1) and need not be given in detail.

Ex. 3. Find the area between the curve $y^3 = kx$, the Y-axis, and the abscissas for $y = 2$ and $y = 3$.

$$A = \int_2^3 x\,dy = \frac{1}{k}\int_2^3 y^3\,dy = \frac{1}{k}\cdot\frac{1}{4}y^4\Big]_2^3 = \frac{65}{4k}.$$

Ex. 4. Find the area included between the hyperbola $xy = 36$ and the straight line $x + y = 15$ (Fig. 57).

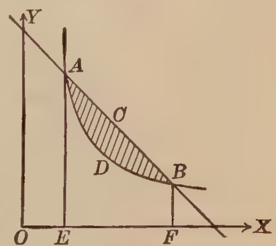

FIG. 57.

The points of intersection of the two curves are found by solving the equations simultaneously. The solution gives (3, 12) and (12, 3) as the coördinates of A and B, respectively.

$$\begin{aligned}\text{Area } ABCD &= \text{area } EACBF - \text{area } EADBF\\ &= \int_3^{12}(15 - x)\,dx - 36\int_3^{12}\frac{dx}{x}\\ &= \left[15x - \frac{x^2}{2} - 36\log x\right]_3^{12} = 17.59.\end{aligned}$$

105. Plane areas, polar coördinates. When polar coördinates are used, the area swept over by the radius vector in passing from an initial position to a final position is the required area.

Let AB, Fig. 58, represent the curve $\rho = f(\theta)$, where $f(\theta)$ is a continuous function. Let OX be the initial line, and α and β the

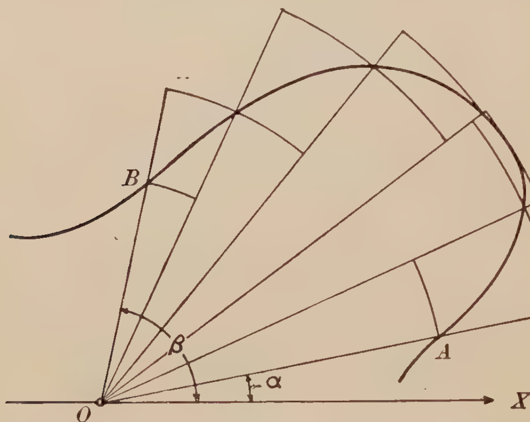

FIG. 58.

initial and final values of θ. The area required is that of the sector AOB.

Let the angle $AOB = \beta - \alpha$ be divided into n parts, $\Delta\theta_1$, $\Delta\theta_2$, etc. In the ith division, let ρ_i' and ρ_i'' be respectively the smallest and largest values of ρ. If from O as a center, arcs are drawn with ρ_i' and ρ_i'' as radii, two circular sectors are formed whose areas are respectively $\frac{1}{2}\rho_i'^2\Delta\theta_i$ and $\frac{1}{2}\rho_i''^2\Delta\theta_i$. Proceeding in this way with each of the n divisions, we get n circular sectors with arcs lying within the given curve AB, and n circular sectors with arcs lying without the curve. The required area OAB is greater than the sum of the inner sectors and less than the sum of the outer sectors; that is,

$$\sum_{\alpha}^{\beta} \tfrac{1}{2}\rho'^2\Delta\theta \leq \text{area } AOB \leqq \sum_{\alpha}^{\beta} \tfrac{1}{2}\rho''^2\Delta\theta. \qquad (1)$$

Since $\sum_{\alpha}^{\beta} \frac{1}{2}\rho'^2\Delta\theta$ and $\sum_{\alpha}^{\beta} \frac{1}{2}\rho''^2\Delta\theta$ always remain finite, the first

never decreasing and the second never increasing, each sum has a limit, and by the method used in Art. 100, it may be shown that these limits are equal. We have, therefore,

$$\text{area } AOB = \underset{\Delta\theta \doteq 0}{L} \sum_{\alpha}^{\beta} \tfrac{1}{2}\rho'^2 \Delta\theta = \underset{\Delta\theta \doteq 0}{L} \sum_{\alpha}^{\beta} \tfrac{1}{2}\rho''^2 \Delta\theta. \qquad (2)$$

Furthermore, since ρ is a continuous function of θ, we may replace the common limit by a definite integral and write

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2 d\theta, \qquad (3)$$

where A denotes the required area.

Ex. 1. Find the area swept over by the radius vector of the curve $\rho = \dfrac{a}{\theta}$, as θ varies from π to 2π.

$$A = \frac{1}{2}\int_{\pi}^{2\pi} \rho^2 d\theta = \frac{a^2}{2}\int_{\pi}^{2\pi} \frac{d\theta}{\theta^2} = \frac{a^2}{4\pi}.$$

Ex. 2. Find the area swept over by the radius vector of the logarithmic spiral $\rho = e^{k\theta}$ in one revolution.

$$A = \frac{1}{2}\int_{0}^{2\pi} e^{2k\theta} d\theta = \frac{1}{4k} e^{2k\theta}\Big]_0^{2\pi} = \frac{1}{4k}(e^{4k\pi} - 1).$$

EXERCISES

1. Find the area between the line $y = 3x + 4$, the X-axis, and the lines $x = 0$, $x = 6$.

2. Find the area between the parabola $y = 5x^2$, the X-axis, and the lines $x = 0$, $x = 4$.

3. Find the area under the curve $5y^2 = x^3$ from the origin to the line $x = 5$.

4. Find the area between the parabola $y = 3x^2$ and the straight line $y = 12x$.

5. Take the arc of the equilateral hyperbola $xy = C$ between the points A and B. Show that the area between this arc and the X-axis is the same as the area between the same arc and the Y-axis.

6. Find the area under the catenary $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ between $x = -m$ and $x = m$.

7. Derive a general expression for the area under the curve $xy^m = C$ between $x = a$ and $x = b$. Discuss the case $m = 1$.

8. Find the area swept over by the radius vector of the spiral of Archimedes $\rho = a\theta$ in one revolution.

9. Find the area of the two loops of the lemniscate $\rho^2 = a^2 \cos 2\theta$. Make use of the symmetry of the curve and take narrowest limits of integration.

10. Find the area swept over in two revolutions by the radius vector of the parabolic spiral $\rho^2 = a^2\theta$.

11. Find the total area bounded by the curve $a^4y^2 + b^2x^4 = a^2b^2x^2$.

SUGGESTION: Trace the curve in order to obtain proper limits of integration.

12. Find the area between the curve $y^2(a^2 - x^2) = a^2x^2$, the asymptote $x = a$, and the X-axis.

13. Show that the area bounded by the spiral $\rho\theta = C$ and two radii ρ_1 and ρ_2 is proportional to $\rho_1 - \rho_2$.

14. Find the area of a loop of the curve $\rho^2 = a^2 \sin n\theta$.

106. Volumes of solids of revolution. A plane area $AEFB$, Fig. 59, bounded by the curve EF, whose equation is $y = f(x)$, the axis of X, and the ordinates AE and BF, rotates about the axis OX and thereby generates a solid of revolution. To derive an expression for the volume of this solid we proceed as follows: Let the interval AB be divided into n parts, Δx_1, Δx_2, Δx_3, etc., and let ordinates be erected at the points of division.

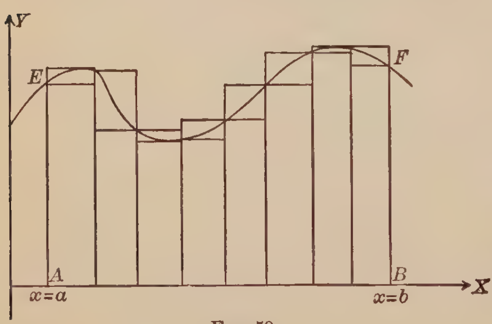

FIG. 59.

In any subinterval Δx_i, let y_i' and y_i'' be respectively the smallest and largest numerical values of the ordinate, and construct rectangles having Δx_i as a base and y_i' and y_i'', respectively, as altitudes. Repeating this construction for each of the subintervals, we obtain one plane area made up of rectangles lying entirely below the given curve EF and a second plane area made up of rectangles whose upper bases lie above this curve. The solids obtained by revolving these areas about the X-axis have respectively the volumes

$$V' = \sum_a^b \pi y'^2 \Delta x, \quad V'' = \sum_a^b \pi y''^2 \Delta x.$$

The volume V of the solid generated by the revolution of the area $AEFB$ must lie between V' and V''. That is,

$$\sum_a^b \pi y'^2 \, \Delta x \leqq V \leqq \sum_a^b \pi y''^2 \, \Delta x. \tag{1}$$

It will be seen that V' and V'' are both monotone functions, the first never decreasing and the second never increasing; hence since the functions are finite, each has a limit as $\Delta x \doteq 0$ (Art. 14). Following the method of Art. 100, it may be shown that the two limits are equal. Consequently we may write

$$V = \underset{\Delta x \doteq 0}{L} \sum_a^b \pi y'^2 \, \Delta x = \underset{\Delta x \doteq 0}{L} \sum_a^b \pi y''^2 \, \Delta x. \tag{2}$$

Since y is taken to be a single-valued and continuous function of x, we may replace the common limit by the definite integral and write

$$\boldsymbol{V = \pi \int_a^b y^2 \, dx.} \tag{3}$$

By a similar process it can be shown that the volume of the solid generated by a rotation about the Y-axis is

$$\boldsymbol{V = \pi \int_c^d x^2 \, dy,} \tag{4}$$

where c and d are the ordinates of the end points of the curve.

Ex. 1. Find the volume generated by the revolution about the X-axis of the area bounded by the line $4x + y = 12$ and the coördinate axes.

For $y = 0$, $x = 3$; hence the limits of integration are $x = 0$, and $x = 3$.

$$V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 (12 - 4x)^2 \, dx = \pi \left(144x - 48x^2 + \frac{16}{3}x^3\right)\Big]_0^3 = 144\pi.$$

Ex. 2. Find the volume generated by the rotation about the X-axis of the area bounded by the segment of the parabola $y^2 = 8x$ between the origin, the X-axis, and the ordinate for $x = 6$.

$$V = \pi \int_0^6 y^2 \, dx = \pi \int_0^6 8x \, dx = 4\pi x^2\Big]_0^6 = 144\pi.$$

The rotation about the Y-axis of the area bounded by the Y-axis, the corresponding abscissa, and the same segment generates the volume

$$V = \pi \int_0^{\sqrt{48}} x^2 \, dy = \pi \int^{\sqrt{48}} \frac{y^4}{64} dy = \frac{\pi y^5}{320}\Big]_0^{\sqrt{48}} = \frac{\pi}{320}(48)^{\frac{5}{2}} = 156.72.$$

EXERCISES

1. A point (m, n) is joined to the origin by a straight line. Find by integration (*a*) the volume of the cone generated by revolving the line about the axis OX; (*b*) about the axis OY.

2. Find the volume of the solid generated by the revolution about the X-axis of a segment of the equilateral hyperbola $xy = 12$ between $x = 4$ and $x = 16$.

3. Find the volume generated by the same segment about the Y-axis.

4. The segment of the curve $y = \sec x$ between $x = 0$ and $x = \frac{1}{4}\pi$ is revolved about the X-axis. Find the volume of the solid generated.

5. Find the volume generated by the revolution of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)$$

about the X-axis, taking $x = -c$ and $x = c$ as limits.

6. Find the volume generated by revolving about the X-axis the part of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ intercepted by the axes.

7. Find the volume of the solid generated by revolving about the X-axis the plane area bounded by the X-axis, the line $x = a$, and the cissoid $y^2 = \dfrac{x^3}{2a - x}$.

8. The area lying to the left of the Y-axis, between the exponential curve $y = e^x$ and the X-axis, is revolved about the X-axis. Find the volume of the solid thus generated.

107. Volumes determined by the summation of thin slices. The method of determining a volume by taking the limit of a number of thin slices, which we have used in finding the volume of revolution, can be extended to other solids. We conceive the solid to be cut by a number of parallel planes usually perpendicular to one of the coördinate axes. In this way, the solid is divided into a number of thin slices lying along the axis in question. Suppose the cutting planes to be chosen perpendicular to the X-axis; the thickness of the slice may be denoted by Δx, and the area of the cross section will be a function of x, say $F(x)$. Then the volume V lies between two sums $V' = \sum_a^b \alpha\, \Delta x$ and $V'' = \sum_a^b \beta\, \Delta x$, where α and β denote respectively the smallest and largest

values of $F(x)$ within the subinterval Δx; and, furthermore, these sums approach the same limit as $\Delta x \doteq 0$. Hence we have

$$V = \underset{\Delta x \doteq 0}{L} \sum_a^b F(x)\,\Delta x = \int_a^b F(x)\,dx. \quad \text{[Art. 100.]} \tag{1}$$

The limits of integration must be so chosen as to include all the slices.

It will be observed that (1) has the same form as the formula for a plane area. The $F(x)$ which gives the area of the cross section is, however, generally different from the $f(x)$ which gives the ordinate of the bounding curve.

Ex. **1.** A circular cylinder of length h is cut by a plane which passes through the diameter of one base and is tangent to the other base. Required the volume of the piece cut from the cylinder.

Taking the axes as shown in Fig. 60, let the solid be divided into slices by planes parallel to the YZ-plane. The sections cut by the planes are evidently similar triangles. For any triangle as ABC, $AB = y$, and $BC = \frac{y}{a}h$, where a is the radius of the base; hence the area is

$$\frac{y^2 h}{2a} = \frac{(a^2 - x^2)\,h}{2a}.$$

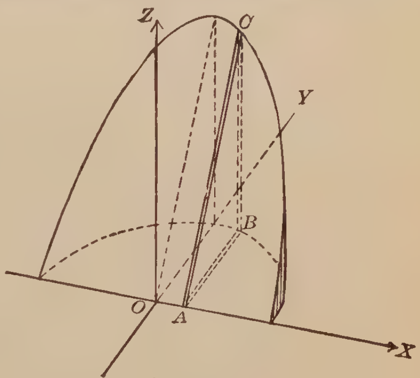

Fig. 60.

For the total volume we have therefore

$$V = \frac{h}{2a} \cdot 2\int_0^a (a^2 - x^2)\,dx = \frac{2}{3}a^2 h.$$

Ex. **2.** Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let the cutting planes be taken parallel to the YZ-plane; then a plane section at a distance x from the YZ-plane will be an ellipse having for its equation

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

This equation may be written

$$\frac{y^2}{\frac{b^2}{a^2}(a^2 - x^2)} + \frac{z^2}{\frac{c^2}{a^2}(a^2 - x^2)} = 1.$$

The semiaxes of the ellipse are therefore $\frac{b}{a}\sqrt{a^2-x^2}$ and $\frac{c}{a}\sqrt{a^2-x^2}$, and the area of the ellipse is $\frac{\pi bc}{a^2}(a^2-x^2)$.

Hence, we have

$$V=\frac{2\,\pi bc}{a^2}\int_0^a(a^2-x^2)\,dx=\frac{4}{3}\pi abc.$$

EXERCISES

1. Find the volume of the elliptic paraboloid $\frac{y^2}{4}+\frac{z^2}{9}=4\,x$ between the planes $x=0$ and $x=3$.

2. Find the volume of the solid bounded by the surface $\frac{x^4}{a^4}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

3. By the method of integration derive the formula for the volume of a pyramid or cone.

4. Derive the formula for the volume of the frustum of a pyramid or cone.

5. A right circular cylinder of base radius a and altitude h has two slices cut from it by planes passing through a diameter of one base and touching the other base. Find the volume of the wedge-shaped solid remaining.

6. A cap for a post is a solid of which every horizontal cross section is a square, and the corners of the squares lie in the surface of a sphere 14 inches in diameter with its center in the upper face of the cap. The depth of the cap is 4 inches. Find the volume.

7. A solid is formed by a variable square whose center moves along the major axis of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ in such a way that the side of the square is always equal to the double ordinate of the ellipse. Find the volume of the solid.

108. Lengths of curves, rectangular coördinates. The length of a curve may be defined as the limit of the sum of the lengths of the inscribed chords as the number of the chords is increased without limit and the length of each approaches the limit zero.

Given the curve AB, Fig. 61, whose equation is $y=f(x)$, where $f(x)$ is a continuous function having a continuous derivative. The length s of this curve between the limits $x=a$, $x=b$ is required. Divide the given interval $b-a$, into n equal parts, denoting by Δx one of these equal divisions. Denote by Δc, Δy, respectively, the corresponding values of the chord and

of the increment of y. From the definition of the length of a curve, we may now write

$$s = \underset{\Delta x \doteq 0}{L} \sum_a^b \Delta c = \underset{\Delta x \doteq 0}{L} \sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x. \tag{1}$$

With the restrictions imposed upon $y = f(x)$, the quotient $\frac{\Delta y}{\Delta x}$, by

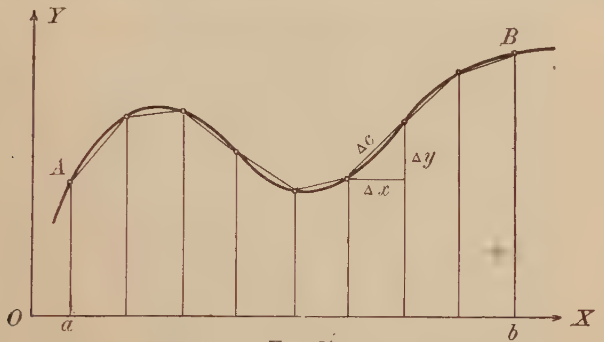

FIG. 61.

the law of the mean, is equal to the derivative $\frac{dy}{dx}$ for some value of x within the subinterval Δx. Hence, since by Art. 100 we may take the value of the integrand $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ at any point within this subinterval, we may in (1) replace $\frac{\Delta y}{\Delta x}$ by $\frac{dy}{dx}$ and write

$$s = \underset{\Delta x \doteq 0}{L} \sum_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x.$$

Replacing the summation by the definite integral, we have finally

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{2}$$

If y is taken as the independent variable, this formula for the length of a curve becomes

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \tag{3}$$

where c, d are the values of y corresponding to $x = a$, $x = b$, respectively.

Ex. 1. Find the length of the curve $y^3 = 3x^2$ from the point (0, 0) to the point (3, 3).

From the given equation we have

$$\frac{dx}{dy} = \frac{y^2}{2x},$$

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{4x^2 + y^4}{4x^2} = \frac{1}{4}(4 + 3y),$$

$$s = \int_0^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \frac{1}{2}\int_0^3 \sqrt{4 + 3y}\, dy = \frac{1}{9}(4 + 3y)^{\frac{3}{2}}\Big]_0^3$$

$$= \tfrac{1}{9}\left(13^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = 4.32.$$

Ex. 2. Find the length of the cycloid described by a point on the circumference of a rolling disk during one revolution of the disk.

The equations of the cycloid are

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta),$$

in which θ denotes the angle through which the disk has turned. Differentiating, we have

$$dx = a(1 - \cos\theta)\, d\theta,$$
$$dy = a \sin\theta\, d\theta,$$

whence $$dx^2 + dy^2 = 2a^2(1 - \cos\theta)\, d\theta^2.$$

From (2), we may write $$s = \int \sqrt{dx^2 + dy^2},$$

whence $$s = \int (dx^2 + dy^2)^{\frac{1}{2}} = a\sqrt{2}\int (1 - \cos\theta)^{\frac{1}{2}}\, d\theta.$$

Now $\sqrt{1 - \cos\theta} = \sqrt{2}\sin\frac{1}{2}\theta$, and the limits of θ are obviously 0 and 2π for one revolution.

Hence $$s = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = -4a\cos\frac{\theta}{2}\Big]_0^{2\pi} = 8a.$$

EXERCISES

1. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

2. Find the length of the semicubical parabola $ay^2 = x^3$ from the origin to the point (m, n).

3. Find the length of the catenary $y = \frac{1}{2}a\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, from $x = 0$ to $x = x_1$.

4. Find the circumference of a circle (*a*) using the equation $x^2 + y^2 = a^2$; (*b*) using the equations $x = a\cos\theta$, $y = a\sin\theta$.

5. The involute of a circle is given by the equations

$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta).$$

Find the length of an arc between the limits $\theta = 0$ and $\theta = \pi$.

109. Lengths of curves, polar coördinates. The formulas of the preceding article for finding the length of a curve in rectangular coördinates may be so transformed as to cover the case of polar coördinates by means of the equations

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta. \tag{1}$$

From Art. 108, we have

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{2}$$

or introducing the differential dx under the radical,

$$s = \int_a^b \sqrt{dx^2 + dy^2}. \tag{3}$$

From equation (1) we obtain

$$\left.\begin{aligned} dx &= -\rho \sin \theta\, d\theta + \cos \theta\, d\rho \\ dy &= \rho \cos \theta\, d\theta + \sin \theta\, d\rho \end{aligned}\right\}. \tag{4}$$

Substituting these expressions in (3), we have

$$\begin{aligned} \sqrt{dx^2 + dy^2} &= \sqrt{[(-\rho \sin \theta\, d\theta + \cos \theta\, d\rho)^2 + (\rho \cos \theta\, d\theta + \sin \theta\, d\rho)^2]} \\ &= \sqrt{\rho^2\, d\theta^2 + d\rho^2}. \end{aligned} \tag{5}$$

Consequently, we have

$$s = \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta, \tag{6}$$

where α, β are the limits of integration corresponding to the limits a, b when written in rectangular coördinates.

When ρ is taken as the independent variable, this formula may be written

$$s = \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2}\, d\rho. \tag{7}$$

Ex. Find the whole length of the cardioid

$$\rho = 2\,a(1 - \cos \theta).$$

Differentiating the equation of the curve, we get

$$\frac{d\rho}{d\theta} = 2\,a \sin \theta.$$

Substituting in (6), we have

$$\begin{aligned} s &= 2\int_0^\pi 2\,a[(1 - \cos \theta)^2 + \sin^2 \theta]^{\frac{1}{2}}\, d\theta \\ &= 2\int_0^\pi 4\,a \sin \frac{\theta}{2}\, d\theta = 16\,a\left[-\cos \frac{\theta}{2}\right]_0^\pi = 16\,a. \end{aligned}$$

EXERCISES

1. Find the length of the circumference of the circle $\rho = 2\,a\cos\theta$.

2. Find the length of the logarithmic spiral $\rho = e^{a\theta}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$; also from $\theta = \frac{\pi}{2}$ to $\theta = \pi$.

3. Find the length of the spiral $\rho = e^{a\theta}$ from the pole ($\rho = 0$) to $\rho \doteq 1$.

4. Find the length of the curve $\rho = a \sec\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{4}$.

110. Areas of surfaces of revolution. Let a plane curve $y = f(x)$ revolve about the X-axis and thereby generate a surface. We may find an expression for the area S of this surface by a method similar to that employed in finding the length of the curve.

Divide the interval AB, Fig. 61, into n equal parts, and denote the length of each by Δx. Denote by Δc the length of the corresponding chord PQ, by Δy the corresponding increment of y, and by $\Delta S'$ the surface generated by revolving the chord Δc about the axis of X. Since the lateral area of the frustum of a cone of revolution is the product of the slant height and one half the sum of the circumferences of the bases, we have for the element of surface thus generated

$$\Delta S' = 2\,\pi\left(y \pm \frac{\Delta y}{2}\right)\Delta c. \tag{1}$$

But
$$\Delta c = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x; \tag{2}$$

hence, taking the sum of all the elements of surface thus formed, we have

$$\sum \Delta S' = \sum 2\,\pi\left(y \pm \frac{\Delta y}{2}\right)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x. \tag{3}$$

As $\Delta x \doteq 0$, the left-hand member of this equation approaches the area of the surface generated by the revolution of the given curve, that is, the required area. In fact, we may define the area of the required surface of revolution as the limit of the sum of the surfaces of these frustums as $\Delta x \doteq 0$. In the right-hand member $\left(y \pm \frac{\Delta y}{2}\right)$ approaches y, and $\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$ becomes $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, since

$y = f(x)$ is assumed to be a continuous function having a continuous derivative. We have then

$$S = \underset{\Delta x \doteq 0}{L} \sum_a^b 2\pi\left(y \pm \frac{\Delta y}{2}\right)\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x, \tag{4}$$

or

$$S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \quad \text{(Arts. 100, 108)} \tag{5}$$

When OY is taken as the axis of revolution, the formula becomes

$$S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \tag{6}$$

From (2) it is evident that we may, if we choose, replace

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

in (5) and (6) by

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

In some problems the latter form is more convenient. The limits of integration must be changed to the values of y corresponding to $x = a$, $x = b$.

Ex. **1**. Find the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis.

From the given equation, we get

$$1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{a}{y}\right)^{\frac{2}{3}}.$$

We have therefore

$$S = 2\cdot 2\pi\int_0^a y\left(\frac{a}{y}\right)^{\frac{1}{3}} dy = 4\pi a^{\frac{1}{3}}\cdot\frac{3}{5}y^{\frac{5}{3}}\Big]_0^a = \frac{12}{5}\pi a^2.$$

Ex. **2**. Find the area of the surface generated by the revolution of a cycloid about its base.

The equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta).$$

Differentiating these equations, we have

$$dx = a(1-\cos\theta)\,d\theta,$$
$$dy = a\sin\theta\,d\theta,$$

whence

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{dx^2+dy^2} = a\sqrt{2(1-\cos\theta)}\,d\theta.$$

Substituting in (5), we get

$$S = 2\pi a^2 \int_0^{2\pi} \sqrt{2(1-\cos\theta)^3}\, d\theta$$
$$= 16\pi a^2 \int_0^{2\pi} \sin^3\left(\frac{\theta}{2}\right) d\left(\frac{\theta}{2}\right) = \frac{64}{3}\pi a^3.$$

EXERCISES

1. Find the area of the surface generated by the revolution of the parabola $y^2 = 8x$ about the X-axis. Take the limits $x = 0$ and $x = 2$; also the limits $x = 2$ and $x = 8$.

2. A line joins the origin to the point (m, n). Find by integration the surface of the cone generated by revolving this line about the X-axis.

3. Find the area generated by revolving about the X-axis the arc of the cubical parabola $2y = x^3$ between $x = 0$ and $x = 2$.

4. Find the area of the surface generated by revolving about the X-axis the arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ between $x = 0$ and $x = a$.

111. Mean value. Let $y = f(x)$ be a continuous function within the interval $x = a$ and $x = b$, and suppose this interval to be divided into n equal parts each equal to Δx. Then $b - a = n\,\Delta x$. Denoting by $y_1, y_2, y_3, \cdots, y_n$ the values of the function corresponding to the values of x at the middle points of these successive subdivisions, let us form the quotient

$$\frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}, \tag{1}$$

which evidently is merely the arithmetic mean of the n values of y. This quotient will vary with the number of divisions n, and the limit which it approaches as n is indefinitely increased is called the **mean value** of the function for the interval $b - a$.

Substituting for n its value $\frac{b-a}{\Delta x}$, (1) may be written

$$\frac{y_1\,\Delta x + y_2\,\Delta x + \cdots + y_n\,\Delta x}{b - a}.$$

The limiting value of this quotient as n becomes infinite, and consequently as $\Delta x \doteq 0$, is

$$\frac{\underset{\Delta x \doteq 0}{L} \sum_a^b y\,\Delta x}{b - a} = \frac{\int_a^b y\,dx}{b - a}. \tag{2}$$

The numerator of (2) is represented by the area under the curve $x = f(x)$ (as $AEFB$, Fig. 62) between the ordinates for $x = a$ and $x = b$. Hence if a rectangle $AMNB$ is constructed having an area equal to the area under the curve, the altitude of this rectangle represents the quotient

$$\frac{\int_a^b y\,dx}{b-a},$$

or the mean value of the function.

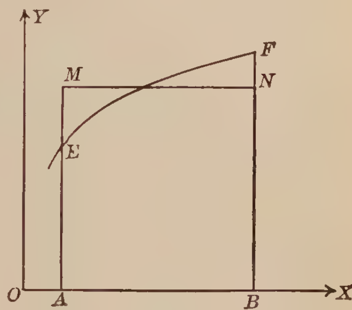

FIG. 62.

The independent variable may be time, distance, angle, area, volume, or any other geometrical or physical magnitude. Mean values may be taken with reference to different variables. Thus in the case of a moving point, values of the velocity may be taken for equal time intervals or for equal space intervals. In the former case, the mean velocity will be the mean ordinate of the velocity curve on a time base; in the latter case, it will be the mean ordinate of the velocity curve on a space base.

EX. 1. In simple harmonic motion the velocity of the moving point is

$$v = a \sin \omega t,$$

and the time of a half-oscillation is $t = \pi/\omega$.

Hence, the mean velocity for the time interval 0 to π/ω is

$$\frac{a\int_0^{\frac{\pi}{\omega}} \sin \omega t\,dt}{\frac{\pi}{\omega} - 0} = \frac{\left[-\frac{a}{\omega}\cos \omega t\right]_0^{\frac{\pi}{\omega}}}{\frac{\pi}{\omega}} = \frac{2}{\pi}a.$$

EX. 2. In the case of a falling body we have

$$v = gt$$

and

$$v^2 = 2\,gs.$$

For the mean velocity, taking time as the variable, we have for the interval 0 to t_1,

$$\frac{\int_0^{t_1} v\,dt}{t_1 - 0} = \frac{g\int_0^{t_1} t\,dt}{t_1} = \frac{1}{2}gt_1.$$

Since the velocity at the end of the time t_1 is gt_1, the mean velocity is one half the final velocity.

The mean velocity for the space s_1, taking equal space intervals, is

$$\frac{\int_0^{s_1} v\,ds}{s_1 - 0} = \frac{\sqrt{2g}\int_0^{s_1}\sqrt{s}\,ds}{s_1} = \frac{2}{3}\sqrt{2gs_1} = \frac{2}{3}v_1;$$

that is, the mean velocity is two thirds the final velocity.

EXERCISES

1. Find the mean of the ordinates of the parabola $y^2 = 10x$ from $x = 0$ to $x = 8$.

2. Find the mean ordinate of the curve $y = x^2 - 7x + 5$ between $x = 1$ and $x = 5$.

3. Find the mean value of the ordinates of the curve $y = \cos x$ (*a*) between $x = 0$ and $x = \frac{1}{2}\pi$; (*b*) between $x = 0$ and $x = \pi$.

4. A number n is divided into two parts; find the mean value of the product of the parts.

5. During the expansion of steam in an engine cylinder the pressure falls approximately according to the law $px = C$, where x denotes the distance the piston has moved from the beginning of the stroke. Derive an expression for the mean pressure exerted during the period of expansion.

6. Find the mean ordinate of a semicircle of radius a provided the ordinates are drawn at equal intervals on the arc.

SUGGESTION: Use polar coördinates.

112. Work of a variable force. When the point of application of a force is moved in the direction of the line of action of the force, the force is said to do **work**. For example, work is done by the drawbar pull of a locomotive when the locomotive moves along the track, thus moving the point of application.

If the force is constant in magnitude, the work done is the product of the force and the distance through which the point of application moves in the direction of the force. If W denotes the work, F the force, and s the displacement of the point of application, then

$$W = Fs. \tag{1}$$

For example, the work done in raising a load of 800 pounds a height of 6 feet is $800 \times 6 = 4800$ foot-pounds.

The magnitude of the force may vary as the point of application moves, as in compressing a spring. In this case, the work may be expressed as a definite integral as follows: Let s_1 and s_2 be the initial and final distances of the point of application of the force from some origin, so that $s_2 - s_1$ is the displacement. Let $s_2 - s_1$ be divided into n subintervals Δs_1, Δs_2, etc., and in any subinterval Δs_i let F_i' and F_i'' be respectively the smallest and largest values of the force F. Then denoting the work by W, we have

$$\sum_{s_1}^{s_2} F' \Delta s \leqq W \leqq \sum_{s_1}^{s_2} F'' \Delta s. \qquad (2)$$

As in Art. 100, we may show that the functions $\sum_{s_1}^{s_2} F'\Delta s$ and $\sum_{s_1}^{s_2} F'' \Delta s$ have the same limit as $\Delta s \doteq 0$, and we may therefore write at once, since F is a continuous function of s,

$$W = \underset{\Delta s \doteq 0}{L} \sum_{s_1}^{s_2} F \Delta s = \int_{s_1}^{s_2} F ds. \qquad (3)$$

From the method of deriving (3), it is clear that the work of a force may be represented by an area. On the displacement S_1S_2 let the forces F corresponding to the successive positions of the point of application be laid off as ordinates, and let a curve be drawn through the ends of these ordinates (see Fig. 63); then the area S_1ABS_2 under this curve will represent the work of the force between S_1 and S_2. For the area under the curve is

$$\int_{s_1}^{s_2} F ds,$$

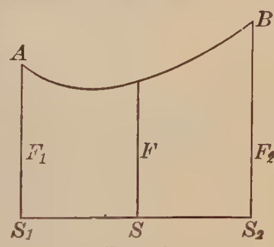

FIG. 63.

which according to (3) gives the work W.

In order to integrate the expression $F ds$, the force F must be expressed as a function of the displacement s. The following cases are those occurring most frequently in practice.

(*a*) *When the force is a linear function of the displacement.*

This is the law in the compression of a spring. We have

$$F = ks + b,$$

$$W = \int_{s_1}^{s_2} (ks + b)\, ds = \frac{1}{2} ks^2 + bs \Big]_{s_1}^{s_2}$$

$$= \frac{1}{2} k (s_2^2 - s_1^2) + b (s_2 - s_1).$$

Ex. **1**. A spring is 10 inches long, and a force of 48 pounds is required for each inch it is compressed. Find the work of compressing the spring from 10 inches to a length of 6 inches; also from a length of 8 inches to a length of 5 inches.

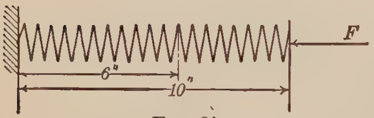

FIG. 64.

In the first case,

$$s_1 = 0,\ s_2 = 10 - 6 = 4,\ F = 48\, s.$$

Hence, $$W = \int_{s_1}^{s_2} F\, ds = 48 \int_0^4 s\, ds = 24\, s^2 \Big]_0^4 = 384 \text{ in. lb.}$$

For the second case, $s_1 = 2,\ s_2 = 5$, and

$$W = 48 \int_2^5 s\, ds = 24\, s^2 \Big]_2^5 = 504 \text{ in. lb.}$$

Ex. **2**. A bar is stretched from its original length L by a gradually increasing load. Denoting by s the amount of stretch for a given force F, Hooke's law gives as the relation between F, s, and L,

$$F = \frac{EAs}{L},$$

where E denotes the coefficient of elasticity of the material, and A the area of the bar. The work of stretching the bar by an amount s is therefore

$$W = \int_0^a F\, ds = \frac{EA}{L} \int_0^a s\, ds = \frac{EAa^2}{2\, L}.$$

We have for wrought iron $E = 30{,}000{,}000$; suppose a bar having a cross-section area of 2 square inches be stretched from 60 inches to 60.5 inches. Here $A = 2,\ a = 0.5$, and $L = 60$; hence the work is

$$W = \frac{30000000 \times 2 \times 0.5^2}{2 \times 60} = 125{,}000 \text{ in. lb.}$$

(*b*) *When the force varies inversely as the displacement.*

In this case,

$$F = \frac{k}{s},$$

whence $$W = \int_{s_1}^{s_2} F\, ds = k \int_{s_1}^{s_2} \frac{ds}{s} = k \log \frac{s_2}{s_1}.$$

(c) *When the force varies inversely as the square of the distance.*

The law of the inverse square applies to gravitational forces, to forces between electric charges, etc.

Ex. Let a positive charge m of electricity be concentrated at a point P, and a unit charge at a point Q at a distance s from P; then the repulsion of charge m on the unit charge is

$$F = \frac{m}{\overline{PQ}^2} = \frac{m}{s^2}.$$

The work required to move the unit charge from $s = a$ to $s = b$ is

$$W = \int_a^b F\,ds = m\int_a^b \frac{ds}{s^2} = -\,m\frac{1}{s}\Big]_a^b = m\left(\frac{1}{a} - \frac{1}{b}\right).$$

113. Work of expanding gases. A gas is confined by the walls of a cylinder and a movable piston, Fig. 65. By virtue of its pressure, the gas expands or increases in volume, moves the piston, and thus does work against an external resistance. Let p denote the pressure exerted by the gas on a unit area (square inch or square foot), and A the area of the piston. Evidently the total force acting on the piston is $F = pA$; and for $F_1\,\Delta s$, $F_2\,\Delta s$, etc., we may write $p_1A\,\Delta s$, $p_2A\,\Delta s$, etc. But $A\,\Delta s$ is the volume swept over by the piston in moving through the distance Δs and may be denoted by Δv; hence $F\,\Delta s = p\,\Delta v$, and the work done is

FIG. 65.

$$W = \underset{\Delta v \doteq 0}{L} \sum_{v_1}^{v_2} p\,\Delta v = \int_{v_1}^{v_2} p\,dv.$$

In using this formula it must be noted (1) that p denotes pressure per unit area, not the total pressure, and (2) that for correct numerical results consistent units must be used, pounds and feet, or pounds and inches throughout; thus, if p is in pounds per square *inch*, v must be in cubic *inches*, and the result will be work in *inch*-pounds.

Ex. 1. Air expands without change of temperature following Boyle's law. $pv = p_1v_1 =$ const. The work of expansion is

$$W = \int_{v_1}^{v_2} p\,dv = p_1v_1 \int_{v_1}^{v_2} \frac{dv}{v} = p_1v_1 \log\frac{v_2}{v_1}.$$

Ex. 2. Air expanding *adiabatically* follows the law

$$pv^k = p_1v_1^k = \text{const., where } k = 1.41.$$

The work done during the expansion is

$$W = \int_{v_1}^{v_2} p\,dv = p_1v_1^k \int_{v_1}^{v_2} v^{-k}\,dv = p_1v_1^k \frac{v^{1-k}}{1-k}\Big]_{v_1}^{v_2}$$

$$= \frac{p_1v_1^k}{k-1}\Big[v_1^{1-k} - v_2^{1-k}\Big] = \frac{p_1v_1 - p_2v_2}{k-1},$$

since $$p_1v_1^k = p_2v_2^k.$$

EXERCISES

1. The length of an unstretched spring is 16 inches, and a force of 225 pounds is required to stretch it 1 inch. Find the work required to stretch it from a length of 18 inches to a length of 22.5 inches.

2. In hoisting coal or ore from a mine, the load consists of two parts: (1) the weight M of the car and contents; (2) the weight of the rope, which is m pounds per foot. Find the work required for hoisting a distance of h feet.

SUGGESTION: Let s denote the distance of the load from the lower level; then $F = M + m(h - s)$, and

$$W = \int_0^h [M + m(h - s)]\,ds.$$

3. In Exs. 1 and 2 draw diagrams showing by areas the work done, and derive the results by elementary geometry.

4. Suppose the force to vary directly as the square of the displacement of its point of application. Derive a formula for the work.

5. Confined air having a volume of 6 cubic feet and a pressure of 80 pounds per square inch expands following the law $pv = \text{const.}$ to a final volume of 20 cubic feet. Find the work done.

6. Use the data of Ex. 5 and find the work done if the expansion is adiabatic, *i.e.* according to the law $pv^{1.41} = \text{const.}$

7. Find the work of stretching a round iron bar having a diameter of 1.5 inches from a length of 40 inches to a length of 42.3 inches. Take $E = 28{,}000{,}000$.

MISCELLANEOUS EXERCISES

1. Find the areas bounded by the following curves, the X-axis, and the ordinates indicated:

(a) $y = x^3 + x + 5$, from $x = 0$ to $x = 6$.
(b) $y = x^3 - 3x^2 + 4$, from $x = 1$ to $x = 5$.
(c) $y = e^x$, from $x = 0$ to $x = 1$.

2. Find the area of one loop of the curve $\rho = a \sin 2\theta$.

3. Find the area between the curve $y = \tan x$ and the X-axis from $x = 0$ to $x = \frac{\pi}{3}$.

4. Find the area between the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes.

5. Find the volume generated by the revolution of the entire curve $x^2 + y^{\frac{2}{3}} = 1$, ($a$) about the X-axis; (b) about the Y-axis.

6. A cylindrical vessel having an altitude of 12 inches and a base diameter of 8 inches is tipped and the contained fluid is poured out until the liquid surface coincides with a diameter of the base. Find the quantity of liquid remaining in the vessel.

7. If e is the eccentricity of an ellipse and ϕ the eccentric angle, the parametric equations of the ellipse are: $x = a \cos\phi$, $y = b \sin\phi$. Show that the entire length of the ellipse is

$$4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\cos^2\phi}\, d\phi.$$

8. The solid shown in Fig. 66 is generated by moving a variable rectangle $DEFG$ parallel to the plane XOY. One angle D moves along the axis OZ, and the other angles E and G move in given curves on the planes YZ and ZX, respectively. If the curves QER and PGR are circular arcs of radius a with centers at O, find the volume of the solid.

9. In Ex. 8 find the convex surface QSR.

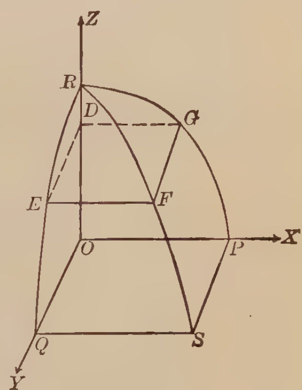

FIG. 66.

10. In Fig. 66, take $OR = 8$, $OP = OQ = 6$, and assume the curves RQ and RP to be parabolic arcs with vertices at R. Find the volume of the solid and the areas of convex surfaces.

11. In Fig. 66, let the curves RP and RQ be elliptic quadrants with major and minor semi-axes of m and n respectively. Find the volume of the solid.

12. The value of a harmonic alternating current is given by the equation $i = i_0 \sin\theta$, where i_0 is the maximum value. Find the mean value of the current for a half cycle, that is, from $\theta = 0$ to $\theta = \pi$.

13. In Fig. 67 is shown a cylindrical journal and bearing. The intensity of the normal pressure at any point P is assumed to be proportional to the depth of P below the diameter AC. If p_0 is the intensity at the lowest point B, find the mean intensity over the surface ABC.

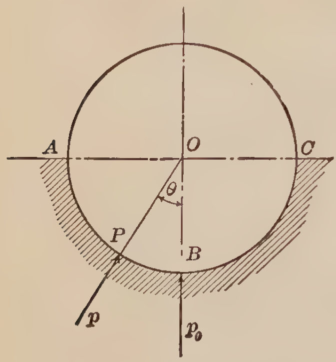

FIG. 67.

14. Find an expression for the work done by a gas expanding isothermally according to van der Waals' equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=C$$

from an initial volume v_1 to a final volume v_2.

15. Find an expression for the work done by a force that varies as the nth power of the displacement of its point of application.

16. Find the work required to compress 20 cubic feet of air from a pressure of 14.5 pounds per square inch to a pressure of 63.5 pounds per square inch, the equation of the compression curve being $pv^{1.3} = \text{const.}$

17. A particle of mass m has simple harmonic motion defined by the equation $x = r\cos\omega t$. Show that the mean kinetic energy ($\frac{1}{2}mv^2$) for a complete oscillation is one half the maximum kinetic energy.

CHAPTER XII

SPECIAL METHODS OF INTEGRATION

114. Integration by parts. Thus far we have made use of only those integrals that could be evaluated by use of the fundamental formulas given in Chapter VI. In some cases we were able to reduce the given function to a fundamental form by a simple transformation. Not all functions, however, can be easily reduced to those types by the methods already employed, and in this chapter we shall consider some special methods by which this reduction may be effected in cases more complicated than those already discussed.

If u and v are two functions of the same independent variable, we have upon differentiating their product,

$$d(uv) = u\,dv + v\,du,$$

or

$$u\,dv = d(uv) - v\,du.$$

Integrating, we get

$$\int u\,dv = uv - \int v\,du. \tag{1}$$

By this formula the integral $\int u\,dv$ is obtained by the evaluation of another integral $\int v\,du$. This method is called **integration by parts**, and is one of the most useful of the integral calculus. It is particularly helpful in the integration of the product of two functions where the integral of one can be easily found; also in the integration of logarithmic functions, exponential functions, and inverse trigonometric functions. No general directions can be given as to which of the two functions is to be taken as u and which as dv, except that the selection should be such as will render dv and $v\,du$ most easily integrable. In case of doubt, first

try putting dv equal to the most complicated factor that can be easily reduced to a fundamental form.

Ex. 1. $\int xe^x\,dx.$

Put $\qquad dv = e^x\,dx, \qquad u = x,$

whence $\qquad v = e^x, \qquad du = dx.$

Substituting these values in formula (1), we have

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1) + C.$$

Ex. 2. $\int x \operatorname{arc}\tan x\,dx.$

Put $\qquad dv = x\,dx, \qquad u = \operatorname{arc}\tan x,$

whence $\qquad v = \frac{1}{2}x^2, \qquad du = \frac{dx}{1+x^2}.$

Substituting these values in the formula, we have

$$\begin{aligned}\int x \operatorname{arc}\tan x\,dx &= \frac{1}{2}x^2 \operatorname{arc}\tan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2 \operatorname{arc}\tan x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{1+x^2} \\ &= \frac{1}{2}x^2 \operatorname{arc}\tan x - \frac{1}{2}x + \frac{1}{2}\operatorname{arc}\tan x + C \\ &= \frac{x^2+1}{2}\operatorname{arc}\tan x - \frac{1}{2}x + C.\end{aligned}$$

Ex. 3. $\int \sqrt{a^2-x^2}\,dx.$

Put $\qquad u = \sqrt{a^2-x^2}, \qquad dv = dx,$

whence $\qquad du = -\frac{x\,dx}{\sqrt{a^2-x^2}}, \qquad v = x.$

The result of substituting these values in (1) is

$$\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + \int\frac{x^2\,dx}{\sqrt{a^2-x^2}}.$$

We may write

$$\frac{x^2}{\sqrt{a^2-x^2}} = -\frac{a^2-x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2} + \frac{a^2}{\sqrt{a^2-x^2}}.$$

We have therefore

$$\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}},$$

whence $\qquad 2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\operatorname{arc}\sin\frac{x}{a},$

and $\qquad \int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\operatorname{arc}\sin\frac{x}{a}\right].$

EXERCISES

1. $\int x^2 e^x \, dx.$
2. $\int x^m \log x \, dx.$
3. $\int \text{arc} \sin \theta \, d\theta.$
4. $\int \text{arc} \cot \theta \, d\theta.$
5. $\int \sin^2 \theta \, d\theta.$
6. $\int e^{ax} \sin bx \, dx.$
7. $\int \log x \, dx.$
8. $\int \cos \theta \log \sin \theta \, d\theta.$
9. $\int x^2 \text{arc} \tan x \, dx.$
10. $\int e^{\frac{x}{a}} \cos \frac{x}{a} \, dx.$
11. $\int x \cos x \, dx.$
12. $\int \sec^3 \theta \, d\theta.$
13. $\int_0^a \sqrt{x^2 + a^2} \, dx.$
14. $\int_0^a x^3 \sqrt{a^2 - x^2} \, dx.$
15. $\int x^3 \log^2 x \, dx.$
16. $\int_0^{\frac{\pi}{2}} \cos \theta \cos 2\theta \, d\theta.$

115. Integration of rational fractions. By a rational fraction we mean the quotient of two rational integral functions. We shall consider only those fractions in which the numerator is of lower degree than the denominator; for, if this is not the case, we can always by division reduce the given expression to a rational integral function plus such a fraction. The decomposition of rational fractions is fully treated in algebra,* and a knowledge of the principles and methods involved is assumed here. In the present article, we shall show how the decomposition of rational fractions may be employed in simplifying an integration. We shall consider the following cases.

(*a*) *When the denominator is the product of several linear factors, none of which is repeated.*

The given function can then be written in the form

$$f(x) = \frac{\phi(x)}{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)}.$$

We may now assume

$$f(x) = \frac{A}{x - \alpha_1} + \frac{B}{x - \alpha_2} + \cdots + \frac{K}{x - \alpha_k},$$

* See Rietz & Crathorne's *College Algebra*, pp. 203–208.

and calculate the numerators A, B, $\cdots$, K by the principles of undetermined coefficients. The integral of $f(x)$ is then found by taking the sum of the integrals of the separate terms. The following examples illustrate the method.

Ex. 1. $\displaystyle\int \frac{(5x+1)\,dx}{x^2-2x-35}.$

We have

$$\frac{5x+1}{x^2-2x-35} = \frac{5x+1}{(x+5)(x-7)} = \frac{A}{x+5} + \frac{B}{x-7}.$$

Clearing of fractions, we obtain

$$5x+1 = A(x-7) + B(x+5).$$

Equating the coefficients of the same powers of x in the two members of this equation, there results

$$5 = A + B,$$

and

$$1 = 5B - 7A,$$

from which

$$A = 2, \quad B = 3.$$

Substituting these values in the original equation, we have

$$\begin{aligned}\int \frac{(5x+1)\,dx}{(x+5)(x-7)} &= \int \frac{2\,dx}{x+5} + \int \frac{3\,dx}{x-7} \\ &= 2\log(x+5) + 3\log(x-7) + C \\ &= \log\left[(x+5)^2(x-7)^3\right] + C.\end{aligned}$$

Ex. 2. $\displaystyle\int \frac{(x^2+x-1)\,dx}{x^3+x^2-6x}.$

Since

$$x^3+x^2-6x = x(x+3)(x-2),$$

we have

$$\frac{x^2+x-1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}.$$

Clearing of fractions, we get the identity

$$x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3).$$

For $x = 0$, we obtain $A = \frac{1}{6}$; for $x = -3$, $B = \frac{1}{3}$; and for $x = 2$, $C = \frac{1}{2}$. Hence, we have

$$\frac{x^2+x-1}{x^3+x^2-6x} = \frac{1}{6x} + \frac{1}{3(x+3)} + \frac{1}{2(x-2)}.$$

Therefore,
$$\begin{aligned}\int \frac{(x^2+x-1)\,dx}{x^3+x^2-6x} &= \int \frac{dx}{6x} + \int \frac{dx}{3(x+3)} + \int \frac{dx}{2(x-2)} \\ &= \tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2) + C \\ &= \log\sqrt[6]{x(x+3)^2(x-2)^3} + C.\end{aligned}$$

EXERCISES

1. $\int \frac{(x+2)\,dx}{x^2-5x+6}$.

2. $\int \frac{(3x-5)\,dx}{x^2+2x-15}$.

3. $\int \frac{(3x+4)\,dx}{x^3-7x^2+12x}$.

4. $\int \frac{(2x-1)\,dx}{x^2+3x+2}$.

5. $\int \frac{dx}{2+7x+3x^2}$.

6. $\int \frac{x^4-5x^3+6x-4}{x^3-4x^2-5x}\,dx$.

7. $\int \frac{(x^2+mn)\,dx}{x(x+m)(x-n)}$.

8. $\int \frac{9x^2-4x-8}{x^3-4x}\,dx$.

(*b*) *When the denominator is the product of several linear factors, some of which are repeated.*

In this case, the given function has the form

$$f(x)=\frac{\phi(x)}{(x-\alpha)(x-\beta)^k\cdots(x-\gamma)^s},$$

which can be written in the form

$$f(x)=\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{C}{(x-\beta)^2}+\cdots+\frac{K}{(x-\beta)^k}+\cdots$$
$$+\frac{L}{x-\gamma}+\frac{M}{(x-\gamma)^2}+\cdots+\frac{S}{(x-\gamma)^s}.$$

We may calculate the coefficients $A, B, C, \cdots, K, L, M, \cdots, S$, as in case ($a$). The following example illustrates this method.

Ex. $\int \frac{(x-8)\,dx}{x^3-4x^2+4x}$.

We have

$$\frac{x-8}{x^3-4x^2+4x}=\frac{x-8}{x(x-2)^2}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}.$$

Clearing of fractions, we get the identity

$$x-8=A(x-2)^2+Bx(x-2)+Cx.$$

Equating the coefficients of like powers of x, we have

$$-8=4A,\qquad 1=-4A-2B+C,\qquad 0=A+B;$$

and from these equations we obtain

$$A=-2,\qquad B=2,\qquad C=-3.$$

Hence
$$\int \frac{(x-8)\,dx}{x^3-4x^2+4x} = -2\int \frac{dx}{x} + 2\int \frac{dx}{x-2} - 3\int \frac{dx}{(x-2)^2}$$
$$= -\log x^2 + \log (x-2)^2 + \frac{3}{x-2} + C$$
$$= \log\left(\frac{x-2}{x}\right)^2 + \frac{3}{x-2} + C.$$

EXERCISES

1. $\int \frac{(x-4)\,dx}{x^3-4x^2+4x}$.
2. $\int \frac{(3x-2)\,dx}{x(x+3)^2}$.
3. $\int \frac{(2x-5)\,dx}{(x-2)^3}$.
4. $\int \frac{(x-1)\,dx}{(x+3)^2}$.
5. $\int \frac{(4y-3)\,dy}{y^3-3y^2}$.
6. $\int \frac{6x^3-8x^2-4x+1}{x^4-2x^3+x^2}\,dx$.
7. $\int \frac{(3x-1)\,dx}{x^3-x^2-x+1}$.
8. $\int \frac{x\,dx}{(x+1)^2(x-1)}$.
9. $\int \frac{x^2+2x-3}{x^3-x^2}\,dx$.
10. $\int \frac{mx^2\,dx}{(m+x)^3}$.

(c) *When the denominator contains factors of the second degree.* Aside from linear factors, the denominator may contain quadratic factors not decomposable into real linear factors. For the linear factors, we assume a decomposition into partial fractions in accordance with the principles of cases (a) and (b). Corresponding to the quadratic factors, we assume fractions whose numerators are linear in the variable. If any of the quadratic factors appear to a degree higher than the first, then we assume in the decomposition as many fractions as the degree of the factor, the numerator of each being linear and the denominator having the given factor in increasing powers. The following examples illustrate the method.

Ex. 1. $\int \frac{x^2\,dx}{(x-1)(x^2+1)}$.

We write
$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Clearing of fractions and equating coefficients of like powers of x, we find
$$A = B = C = \tfrac{1}{2}.$$

Hence
$$\int \frac{x^2\,dx}{(x-1)(x^2+1)} = \frac{1}{2}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{x+1}{x^2+1}\,dx$$
$$= \tfrac{1}{2}\log(x-1) + \tfrac{1}{4}\log(x^2+1) + \tfrac{1}{2}\arctan x + C$$
$$= \tfrac{1}{4}\log(x-1)^2(x^2+1) + \tfrac{1}{2}\arctan x + C.$$

Ex. 2. $\displaystyle\int \frac{x+1}{(x-1)(x^2+1)^2}\,dx.$

If we write
$$\frac{x+1}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$
we find by clearing of fractions and equating like powers of x the following values for the coefficients:
$$A = \tfrac{1}{2},\ B = -\tfrac{1}{2},\ C = -\tfrac{1}{2},\ D = -1,\ E = 0.$$
Hence
$$\int \frac{x+1}{(x-1)(x^2+1)^2}\,dx = \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{x+1}{x^2+1}\,dx - \int\frac{x\,dx}{(x^2+1)^2}$$
$$= \tfrac{1}{2}\log(x-1) - \tfrac{1}{4}\log(x^2+1) - \tfrac{1}{2}\arctan x + \frac{1}{2(x^2+1)} + C.$$

EXERCISES

1. $\displaystyle\int \frac{5x^2+10x-3}{x^4-1}\,dx.$
2. $\displaystyle\int \frac{dx}{x^4-1}.$
3. $\displaystyle\int \frac{(2x^4-1)\,dx}{x^2(1+x^2)^2}.$
4. $\displaystyle\int \frac{x\,dx}{x^4+4x^2+3}.$
5. $\displaystyle\int \frac{x\,dx}{x^4+5x^2+4}.$
6. $\displaystyle\int \frac{(z^2-1)\,dz}{z^4+2z^2+1}.$
7. $\displaystyle\int \frac{(y^3-4)\,dy}{y^4+2y^2-3}.$
8. $\displaystyle\int \frac{dx}{(x+a)(x^2+b^2)}.$
9. $\displaystyle\int \frac{dx}{x^2(x^2+3)}.$
10. $\displaystyle\int \frac{x\,dx}{x^3+x^2+x+1}.$
11. $\displaystyle\int \frac{(2x^2-1)\,dx}{(1+x)^2(1+x+x^2)}.$
12. $\displaystyle\int \frac{dx}{x(1+2x^2)}.$
13. $\displaystyle\int \frac{x^3\,dx}{(x^2+1)^2}.$

116. Integration of functions containing radicals. In the present article we shall discuss some special methods by which functions containing radicals may frequently be changed into equivalent functions free from radicals.

(a) *When $f(x)$ contains fractional powers of $a+bx$, but no other radicals.*

In this case, $f(x)$ can be transformed into a rational expression by the substitution

$$a + bx = z^n, \tag{1}$$

where n is the least common multiple of the denominators of all the fractional exponents of $a + bx$. This follows from the fact that $f(x)$ and dx can then be expressed rationally in terms of z and dz, as the following examples will illustrate.

Ex. 1. Find $\displaystyle\int \frac{y\,dy}{(1+y)^{\frac{1}{3}}}$.

Put $$1 + y = z^3,$$

whence $$y = z^3 - 1, \quad dy = 3\,z^2 dz.$$

We have then
$$\begin{aligned}\int \frac{y\,dy}{(1+y)^{\frac{1}{3}}} &= \int \frac{(z^3-1)3\,z^2dz}{z} = 3\int (z^4 - z)\,dz \\ &= 3\int z^4dz - 3\int z\,dz \\ &= \tfrac{3}{5}\,z^5 - \tfrac{3}{2}\,z^2 + C \\ &= \tfrac{3}{5}(y+1)^{\frac{5}{3}} - \tfrac{3}{2}(y+1)^{\frac{2}{3}} + C.\end{aligned}$$

Ex. 2. Find $\displaystyle\int \frac{(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 4)\,dx}{x^{\frac{1}{2}} + 1}$.

Put $x = z^4$, whence $dx = 4\,z^3dz$.
We have then

$$\begin{aligned}\int \frac{(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 4)\,dx}{x^{\frac{1}{2}} + 1} &= \int \frac{(z^2 + z + 4)4\,z^3\,dz}{z^2+1} \\ &= 4\int \left(z^3 + z^2 + 3\,z - 1 - \frac{3\,z-1}{z^2+1}\right) dz\end{aligned}$$
$$\begin{aligned}&= 4\left[\int z^3\,dz + \int z^2\,dz + 3\int z\,dz - \int dz - 3\int \frac{z\,dz}{z^2+1} + \int \frac{dz}{z^2+1}\right] \\ &= 4\left[\tfrac{1}{4}\,z^4 + \tfrac{1}{3}\,z^3 + \tfrac{3}{2}\,z^2 - z - \tfrac{3}{2}\log(z^2+1) + \text{arc tan}\,z + C\right] \\ &= z^4 + \tfrac{4}{3}\,z^3 + 6\,z^2 - 4\,z - 6\log(z^2+1) + 4\,\text{arc tan}\,z + C' \\ &= x + \tfrac{4}{3}\,x^{\frac{3}{4}} + 6\,x^{\frac{1}{2}} - 4\,x^{\frac{1}{4}} - 6\log(x^{\frac{1}{2}}+1) + 4\,\text{arc tan}\,x^{\frac{1}{4}} + C'.\end{aligned}$$

(b) *When $f(x)$ contains the radical $\sqrt{x^2 + ax + b}$ and no other.*

In many cases the integral may be made to depend upon one of the fundamental integrals by writing the radical in the form $\sqrt{u^2 \pm c^2}$. If this method does not lead to a convenient way of per-

forming the integration, we may rationalize $f(x)\,dx$ by the substitution

$$\sqrt{x^2+ax+b}=z-x. \tag{2}$$

Squaring (2), we have

$$x^2+ax+b=z^2-2\,zx+x^2,$$

whence

$$x=\frac{z^2-b}{a+2\,z};$$

$$dx=\frac{2\,(z^2+az+b)}{(a+2\,z)^2}\,dz.$$

Consequently, $f(x)\,dx$ becomes free from radicals upon substituting these values of $\sqrt{x^2+ax+b}$, x, and dx.

Ex. $\displaystyle\int\frac{x+\sqrt{x^2+x+1}}{2\sqrt{x^2+x+1}}\,dx.$

$$\int\frac{x+\sqrt{x^2+x+1}}{2\sqrt{x^2+x+1}}\,dx=\int\frac{x\,dx}{\sqrt{(2\,x+1)^2+3}}+\frac{1}{2}\int dx.$$

Putting $2\,x+1=u$, we have for the first integral

$$\int\frac{x\,dx}{\sqrt{(2\,x+1)^2+3}}=\frac{1}{4}\int\frac{u\,du}{\sqrt{u^2+3}}-\frac{1}{4}\int\frac{du}{\sqrt{u^2+3}}$$
$$=\tfrac{1}{4}\sqrt{u^2+3}-\tfrac{1}{4}\log\left[u+\sqrt{u^2+3}\right]+C.$$

Replacing the value of u and combining, we obtain as the final result

$$\tfrac{1}{2}\left[x+\sqrt{x^2+x+1}\right]-\tfrac{1}{4}\log\left[2\,x+1+2\sqrt{x^2+x+1}\right]+C.$$

(c) *When $f(x)$ contains the radical $\sqrt{-x^2+ax+b}$ and no other.*

We shall consider only those cases in which the expression under the radical can be broken up into real linear factors. The integration can frequently be most readily performed by putting the radical in the form $\sqrt{c^2-u^2}$ and making use of one of the standard forms. If this is not feasible, we may proceed as follows.

Write the radical in the form

$$\sqrt{-x^2+ax+b}=\sqrt{(x-\alpha)(\beta-x)},$$

and $f(x)\,dx$ can be rationalized by the substitution

$$\sqrt{(x-\alpha)(\beta-x)}=(\beta-x)\,z\ \left[\text{or } (x-\alpha)\,z\right],$$

as we shall now show. Squaring, we obtain

$$(x-\alpha)(\beta-x)=(\beta-x)^2z^2,$$

whence $$x=\frac{\beta z^2+\alpha}{1+z^2},\ dx=\frac{(\beta-\alpha)2z}{(1+z^2)^2}dz.$$

The integrand which results from the substitutions is rational.

Ex. Find $\displaystyle\int\frac{dx}{x\sqrt{-x^2+4x-3}}$.

We have here $$\sqrt{(x-1)(3-x)}=(3-x)z,$$

whence $$x=\frac{3z^2+1}{1+z^2},\quad dx=\frac{4z\,dz}{(1+z^2)^2}.$$

Substituting these values in the given integrand, we have

$$\int\frac{dx}{x\sqrt{-x^2+4x-3}}=2\int\frac{dz}{3z^2+1}=\frac{2}{\sqrt{3}}\arctan\sqrt{3}\,z+C$$

$$=\frac{2}{\sqrt{3}}\arctan\sqrt{\frac{3(x-1)}{3-x}}+C.$$

(*d*) *Integration by trigonometric substitution.*

If $f(x)$ contains a radical of the form $\sqrt{a^2\pm u^2}$, where u is some function of x, it can often be easily transformed to an equivalent function free from radicals by the substitution of a trigonometric function. All that is necessary is to substitute for u that trigonometric function which renders the expression under the radical a perfect square. It will be readily seen that this end is accomplished by the following substitutions:

(1) Put $u=a\sin\theta$, or $a\cos\theta$ in functions involving $\sqrt{a^2-u^2}$.

(2) Put $u=a\tan\theta$, or $a\cot\theta$ in functions involving $\sqrt{a^2+u^2}$.

(3) Put $u=a\sec\theta$, or $a\csc\theta$ in functions involving $\sqrt{u^2-a^2}$.

Whenever the resulting trigonometric integrand does not fall at once under one of the fundamental formulas, or take one of the special forms discussed in the following article, the student should apply the methods of integration discussed in the preceding articles.

The following examples illustrate the use of these substitutions.

Ex. 1. $\displaystyle\int\sqrt{a^2-x^2}\,dx.$

Put $x=a\sin z,\ dx=a\cos z\,dz.$

We have then

$$\begin{aligned}\int \sqrt{a^2 - x^2}\,dx &= a^2 \int \cos^2 z\,dz = \tfrac{1}{2}\,a^2 \int (1 + \cos 2\,z)\,dz \\ &= \tfrac{1}{2}\,a^2 \int dz + \tfrac{1}{4}\,a^2 \int \cos 2\,z\,d(2\,z) \\ &= \tfrac{1}{2}\,a^2 z + \tfrac{1}{4}\,a^2 \sin 2\,z + C \\ &= \tfrac{1}{2}\,a^2 \arcsin \frac{x}{a} + \tfrac{1}{2}\,x\sqrt{a^2 - x^2} + C.\end{aligned}$$

Ex. 2. $\displaystyle\int \frac{dx}{x^2\,(x^2 - a^2)^{\frac{1}{2}}}.$

Put $x = a \sec z$, $dx = a \sec z \tan z\,dz$.

Then we have

$$\begin{aligned}\int \frac{dx}{x^2\,(x^2 - a^2)^{\frac{1}{2}}} &= \int \frac{a \sec z \tan z\,dz}{a^2 \sec^2 z\,(a^2 \sec^2 z - a^2)^{\frac{1}{2}}} \\ &= \frac{1}{a^2}\int \cos z\,dz = \frac{\sin z}{a^2} + C = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C.\end{aligned}$$

EXERCISES

1. $\displaystyle\int \frac{x}{1 + x^{\frac{1}{3}}}\,dx.$

2. $\displaystyle\int \frac{2\sqrt{x}\,dx}{4\sqrt{x} - 3\,x^{\frac{2}{3}}}.$

3. $\displaystyle\int x^2\,(mx + b)^{\frac{1}{2}}\,dx.$

4. $\displaystyle\int \frac{dx}{\sqrt{4\,x - 3 - x^2}}.$

5. $\displaystyle\int \frac{3\,dx}{\sqrt{7\,x - 10 - x^2}}.$

6. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 5\,x - 3}}.$

7. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 7\,x + 4}}.$

8. $\displaystyle\int \frac{dx}{x\sqrt{-x^2 + 5\,x - 6}}.$

9. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x - 1}}.$

10. $\displaystyle\int \frac{(3\,x - 1)\,dx}{\sqrt{x^2 - 3}}.$

11. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2\,x + 5}}.$

12. $\displaystyle\int \frac{x^3\,dx}{\sqrt{1 - x^2}}.$

13. $\displaystyle\int \frac{x^3\,dx}{(1 + x^2)^{\frac{3}{2}}}.$

14. $\displaystyle\int \frac{\sqrt{2\,x - x^2}}{x^2}\,dx.$

15. $\displaystyle\int \frac{dx}{\sqrt{x^2 + x}}.$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$

17. $\displaystyle\int \frac{x^2\,dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

18. $\displaystyle\int \frac{x^2\,dx}{(x^2 - a^2)^{\frac{3}{2}}}.$

19. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2 - a^2}}.$

20. $\displaystyle\int (x^2 - a^2)^{\frac{3}{2}}\,dx.$

21. $\int \frac{x^4\,dx}{\sqrt{1-x^2}}$. **22.** $\int \frac{x^3\,dx}{\sqrt{1+x^2}}$.

23. $\int \frac{dx}{x^4\sqrt{x^2-1}}$.

24. $\int_0^a \sqrt{2\,ax - x^2}\,dx$. [Put $x = a\,(1+\sin\theta)$.]

25. $\int_1^4 \frac{dx}{x^4\sqrt{x^2-1}}$. [Put $x = \sec\theta$.]

26. Show that by the substitution of $x = a\tan\theta$ the integral $\int \frac{dx}{(a^2+x^2)^n}$ is reduced to an integral of the form $\int \cos^p\theta\,d\theta$.

27. Integrate Nos. 5, 6, 7, 11, 15 without rationalization.

Derive the following integrals.

28. $\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2}{15\,b^3}(8\,a^2 - 4\,abx + 3\,b^2x^2)\sqrt{a+bx}.$

29. $\int x\sqrt{a+bx}\,dx = -\frac{2}{15\,b^2}(2\,a - 3\,bx)(a+bx)^{\frac{3}{2}}.$

30. $\int x^2\sqrt{a+bx}\,dx = \frac{2}{105\,b^3}(8\,a^2 - 12\,abx + 15\,b^2x^2)(a+bx)^{\frac{3}{2}}.$

117. Integration of special trigonometric functions. There are certain types of trigonometric functions that are readily integrated by easy reductions to standard forms. The following are of this kind:

(*a*) $\int \sec^{2n} x\,dx, \qquad \int \csc^{2n} x\,dx.$

Here n is assumed to be a positive integer. By the substitution $\sec^2 x = 1 + \tan^2 x$, $\sec^{2n} x$ becomes $(1+\tan^2 x)^{n-1}\sec^2 x$; hence, we have

$$\int \sec^{2n} x\,dx = \int (1+\tan^2 x)^{n-1}\,d\,(\tan x).$$

Similarly,

$$\int \csc^{2n} x\,dx = -\int (1+\cot^2 x)^{n-1}\,d\,(\cot x).$$

In each case the binomial can be expanded, and the separate terms of the integrand are then standard forms.

Ex. 1.
$$\begin{aligned}\int \sec^4 x\,dx &= \int (1+\tan^2 x)\sec^2 x\,dx\\ &= \int \sec^2 x\,dx + \int \tan^2 x\sec^2 x\,dx\\ &= \tan x + \tfrac{1}{3}\tan^3 x + C.\end{aligned}$$

Evidently integrals of the form

$$\int \tan^m x \sec^{2n} x\,dx, \qquad \int \cot^m x \csc^{2n} x\,dx$$

can be integrated equally well by this device.

(*b*) $\int \sec^m x \tan^{2n+1} x\,dx, \qquad \int \csc^m x \cot^{2n+1} x\,dx.$

Here n is a positive integer (or zero), and m is any number. We have

$$\int \sec^m x \tan^{2n+1} x\,dx = \int \sec^{m-1} x \cdot (\sec^2 x - 1)^n \sec x \tan x\,dx$$

$$= \int \sec^{m-1} x \cdot (\sec^2 x - 1)^n\, d(\sec x),$$

a form that is readily integrated. A similar form may be deduced for the second integral.

Ex. 2. $\int \sec^4 x \tan^3 x\,dx = \int \sec^3 x \tan^2 x \tan x \sec x\,dx$

$$= \int \sec^3 x(\sec^2 x - 1)\,d(\sec x)$$

$$= \int \sec^5 x\,d(\sec x) - \int \sec^3 x\,d(\sec x)$$

$$= \tfrac{1}{6}\sec^6 x - \tfrac{1}{4}\sec^4 x + C.$$

(*c*) $\int \sin^{2n+1} x\,dx, \qquad \int \cos^{2n+1} x\,dx.$

Again n is assumed to be a positive integer. We have in this case,

$$\int \sin^{2n+1} x\,dx = \int (1 - \cos^2 x)^n \sin x\,dx = -\int (1 - \cos^2 x)^n\,d(\cos x).$$

$$\int \cos^{2n+1} x\,dx = \int (1 - \sin^2 x)^n \cos x\,dx = \int (1 - \sin^2 x)^n\,d(\sin x).$$

Either of these may now be integrated by expanding the parenthesis and integrating the result term by term.

Ex. 3. $\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$

$$= -\int (1 - \cos^2 x)^2\,d(\cos x)$$

$$= -\int d(\cos x) + 2\int \cos^2 x\,d(\cos x) - \int \cos^4 x\,d(\cos x)$$

$$= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.$$

(d) $\int \sin^{2n} x\,dx, \qquad \int \cos^{2n} x\,dx.$

We may perform the integration in this case by the use of multiple angles. From trigonometry we have

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$
$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

By the aid of these formulas the integrand can be expressed in terms of multiple angles of the variable, where the sine and cosine appear only to the first degree.

Ex. **4.** $\int \cos^4 x\,dx.$

We may write this as follows:

$$\int (\cos^2 x)^2\,dx = \tfrac{1}{4}\int (1 + \cos 2x)^2\,dx = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\int (1 + \cos 4x)\,dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + C$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

(e) $\int \sin^m x \cos^n x\,dx.$

There are three cases that demand consideration.

(1) *When either m or n is a positive odd integer.*

In this case, we can reduce the integral to the fundamental form $\int u^n du$. For example, suppose m is odd, say equal to $2k+1$. The given integral may then be written

$$\int \cos^n x \sin^{2k} x \sin x\,dx = -\int \cos^n x (1 - \cos^2 x)^k d(\cos x),$$

which may be easily integrated by expanding the parenthesis and integrating the result term by term.

(2) *When $m + n$ is a negative even integer.*

In this case, we may write

$$\int \sin^m x \cos^n x\,dx = \int \frac{\sin^m x}{\cos^m x}\cos^{m+n} x\,dx$$
$$= \int \tan^m x \sec^{-(m+n)} x\,dx.$$

Since $-(m+n)$ is by hypothesis a positive even integer, the integral falls under type (a).

Ex. 5. $\int \frac{\sin^2 x}{\cos^4 x} dx = \int \tan^2 x \sec^2 x \, dx = \frac{1}{3} \tan^3 x + C.$

(3) *When m and n are positive and both even.*

We may then reduce the given integral to an integral depending upon the integration of the sine of some multiple of the given angle. This transformation may be accomplished by the aid of the substitution $\sin 2x = 2 \sin x \cos x$
and those given in (d).

EXERCISES

1. $\int \sec^6 x \tan x \, dx.$
2. $\int \csc^4 x \, dx.$
3. $\int \cot^2 x \csc^4 x \, dx.$
4. $\int \frac{\cos^2 x}{\sin^4 x} dx.$
5. $\int \sin^3 x \cos^2 x \, dx.$
6. $\int \frac{dx}{\sin^3 x \cos x}.$
7. $\int \tan^3 x \, dx.$
8. $\int \sec^2 x \tan^5 x \, dx.$
9. $\int \sin x \cot^3 x \, dx.$
10. $\int \frac{\sqrt{\sin x}}{\cos^{\frac{5}{2}} x} dx.$
11. $\int \sin^{\frac{1}{3}} x \cos^3 x \, dx.$
12. $\int \cot^2 x \sec^4 x \, dx.$

118. Use of a table of integrals. The process of integration is much more involved and difficult than that of differentiation. It is not possible to integrate all functions in terms of elementary functions, and frequently when integration is possible it is not easily effected. Further consideration of the more complicated cases is, however, rendered unnecessary in an elementary text because of the existence of tables of integrals arranged for convenient reference. Students should become familiar with such a table as a labor-saving device.* The table should not be employed, however, until the student is thoroughly familiar with the various elementary processes of integration. The following examples will illustrate the use of such tables.

Ex. 1. Find the area of a surface of revolution given by the definite integral $2\pi a \int_0^a \frac{x \, dx}{\sqrt{2ax - x^2}}.$

---

* A good table for this purpose is B. O. Pierce's *Short Table of Integrals*, published by Ginn & Co. The student is advised to provide himself with such a table.

From a table of integrals, we find

$$\int \frac{x\,dx}{\sqrt{2\,ax - x^2}} = -\sqrt{2\,ax - x^2} + a \operatorname{arc} \sin \frac{x-a}{a};$$

hence, the required area is

$$2\,\pi a \left\{ -\sqrt{2\,ax - x^2}\Big]_0^a + a \operatorname{arc} \sin \frac{x-a}{a}\Big]_0^a \right\} = \pi a^2(\pi - 2).$$

Ex. **2.** The tractrix is a curve having a constant length of tangent. Required the equation of this curve.

Denoting by a the length of the tangent, we have (Art. 39) $a = y \csc \phi$,

whence $\tan \phi = \dfrac{dy}{dx} = \pm \dfrac{y}{\sqrt{a^2 - y^2}}$, and $\pm\, dx = \dfrac{\sqrt{a^2 - y^2}\,dy}{y}$.

From the table of integrals,

$$\int \frac{\sqrt{a^2 - y^2}\,dy}{y} = \sqrt{a^2 - y^2} - a \log \frac{a + \sqrt{a^2 - y^2}}{y};$$

hence, the required equation is

$$\pm\, x + c = \sqrt{a^2 - y^2} - a \log \frac{a + \sqrt{a^2 - y^2}}{y}.$$

Ex. **3.** Evaluate $\displaystyle\int \frac{d\theta}{\sin^4 \theta}$.

From the table of integrals, we find

$$\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\,\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} x}.$$

Hence $$\int \frac{d\theta}{\sin^4 \theta} = -\frac{1}{3}\,\frac{\cos \theta}{\sin^3 \theta} + \frac{2}{3} \int \frac{d\theta}{\sin^2 \theta}.$$

But $$\int \frac{d\theta}{\sin^2 \theta} = \int \csc^2 \theta\, d\theta = -\cot \theta.$$

The required integral is, therefore,

$$-\frac{1}{3}\left(\frac{\cos \theta}{\sin^3 \theta} + 2 \cot \theta\right) = -\frac{1}{3} \cot \theta\,(\csc^2 \theta + 2).$$

EXERCISES

Evaluate the following integrals by reference to a table of integrals:

1. $\displaystyle\int e^{-2x} \sin x\, dx.$

2. $\displaystyle\int \frac{dx}{x(x^2 - a^2)^{\frac{3}{2}}}.$

3. $\displaystyle\int_0^a (a^2 - x^2)^{\frac{3}{2}}\, dx.$

4. $\displaystyle\int \sin^4 \theta \cos^2 \theta\, d\theta.$

5. $\int_0^a \frac{x^2\,dx}{\sqrt{2\,ax - x^2}}$.

6. $\int \frac{dx}{3 + 2\cos x}$.

7. $\int \frac{dx}{(1 + 2\,x^2)^3}$.

8. $\int \frac{dx}{x^2(3 - 2\,x)}$.

9. $\int \frac{dx}{(4 - 3\,x + x^2)^3}$.

10. $\int \frac{x\,dx}{(x^2 + 2\,x - 5)^2}$.

11. $\int \frac{\sin x\,dx}{x^3}$.

12. $\int_0^{\frac{\pi}{2}} \theta \sin\theta \cos\theta\,d\theta$.

13. $\int_0^{\frac{\pi}{2}} \sin^7\theta\,d\theta$.

14. $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + 2\sin x}$.

15. $\int \frac{x^2\,dx}{\sqrt{x^2 - 3\,x + 7}}$.

16. $\int \sqrt{-x^2 + 6\,x - 1}\,dx$.

17. $\int \frac{\sqrt{1 - x}}{x}\,dx$.

18. $\int \frac{dx}{\sqrt{5 - 4\,x + 2\,x^2}}$.

19. $\int \frac{dx}{(5 - 4\,x + 2\,x^2)^{\frac{3}{2}}}$.

20. $\int \sin^2\theta \cos^2\theta\,d\theta$.

119. Approximate determination of integrals. As stated in Art. 102, the definite integral $\int_a^b f(x)\,dx$ is represented graphically by the area between the curve $y = f(x)$, the X-axis, and the ordinates corresponding to $x = a$ and $x = b$. Ordinarily, the definite integral is evaluated by means of the anti-derivative of $f(x)$. If, however, it is impossible or inconvenient to find the anti-derivative, the definite integral may be evaluated by the following method:

The curve $y = f(x)$ is plotted from $x = a$ to $x = b$, and the area between the curve, the X-axis, and the end ordinates is determined approximately by one of the methods described in the following articles. The numerical measure of the area gives approximately the value of $\int_a^b f(x)\,dx$. Since, as has been shown, the definite integral may represent any measurable magnitude, as area, volume, length, force, quantity of heat, etc., these methods may obviously be used for the approximate determination of any such magnitude.

120. Simpson's rules. The problem of finding any plane area reduces to the problem of finding the area between a plane curve, an assumed X-axis, and two end ordinates. Thus in Fig. 68, the area of the closed figure $AMBNA$ is found by subtracting the area between the curve ANB and OX from the area between AMB and OX.

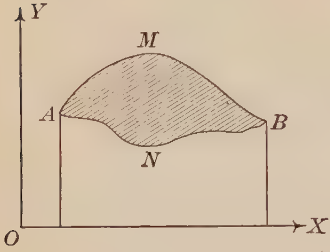

FIG. 68.

Let ACE, Fig. 69, be a part of a curve, and suppose the area between it and the line NR is required, the end ordinates being AN and ER. Let the distance NR be divided into a number of equal parts, each equal to h, and through the points of division let ordinates y_1, y_2, y_3, etc., be drawn. Consider now the three points of the curve, A, B, and C. Taking OB as the Y-axis and O as the origin, the coördinates of these points are

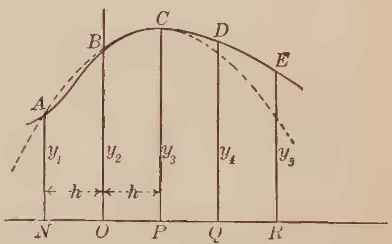

FIG. 69.

$$A \equiv (-h, y_1),\ B \equiv (0, y_2),\ C \equiv (h, y_3). \tag{1}$$

We now assume that the equation of the part of the curve ABC is

$$y = a_0 + a_1 x + a_2 x^2. \tag{2}$$

This is equivalent to the substitution of an arc of a parabola with a vertical axis for the actual curve. With this assumption, the area $NACP$ is

$$\int_{-h}^{h} y\,dx = \int_{-h}^{h} (a_0 + a_1 x + a_2 x^2)\,dx = \frac{h}{3}(6\,a_0 + 2\,a_2 h^2). \tag{3}$$

To determine the coefficients a_0 and a_2, we substitute the coördinates given by (1) in (2). We thus obtain

$$\left.\begin{aligned} y_1 &= a_0 - a_1 h + a_2 h^2,\\ y_2 &= a_0,\\ y_3 &= a_0 + a_1 h + a_2 h^2. \end{aligned}\right\} \tag{4}$$

Solving for a_0 and a_2, we get

$$a_0 = y_2, \quad a_2 = \frac{y_3 + y_1 - 2\,y_2}{2\,h^2},$$

and substituting these values in (3), we have finally

$$\text{area } NABCP = \frac{h}{3}(y_1 + 4\,y_2 + y_3). \tag{5}$$

If now we assume the part of the curve CDE replaced by a second parabolic arc, we get in the same way,

$$\text{area } PCER = \frac{h}{3}(y_3 + 4\,y_4 + y_5),$$

and thus

$$\text{area } NAER = \frac{h}{3}(y_1 + 4\,y_2 + 2\,y_3 + 4\,y_4 + y_5).$$

This process may be repeated any number of times; hence, taking an odd number of ordinates $n+1$, dividing the figure into n strips of equal width h, the approximate area is given by the formula

$$\boldsymbol{A = \frac{h}{3}(y_1 + 4\,y_2 + 2\,y_3 + \cdots + 2\,y_{n-1} + 4\,y_n + y_{n+1})}. \tag{6}$$

This result gives the following rule, known as **Simpson's one-third rule**:

Take the sum of the end ordinates, twice the sum of the intervening odd ordinates, and four times the sum of the even ordinates. Multiply the aggregate by one third of the common distance between the ordinates.

If we take four ordinates y_1, y_2, y_3, y_4, including three spaces, we may pass through their ends A, B, C, and D, a third degree parabola

$$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

Proceeding as before to determine the coefficients, we obtain

$$\text{area } NADQ = \frac{3\,h}{8}(y_1 + 3\,y_2 + 3\,y_3 + y_4). \tag{7}$$

If therefore we divide the whole base into some number of parts n divisible by 3, and take the space in groups of three, or the ordi-

nates in groups of four, we shall have for the successive partial areas,

$$A_1 = \frac{3h}{8}(y_1 + 3y_2 + 3y_3 + y_4),$$

$$A_2 = \frac{3h}{8}(\qquad\qquad\qquad y_4 + 3y_5 + 3y_6 + y_7),$$

$$A_3 = \frac{3h}{8}(\qquad\qquad\qquad\qquad\qquad\qquad y_7 + 3y_8 + 3y_9 + y_{10}),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Adding, we get for the total area,

$$A = \frac{3h}{8}(y_1 + 3y_2 + 3y_3 + 2y_4 + 3y_5 + \cdots + 3y_n + y_{n+1}). \qquad (8)$$

This formula expresses **Simpson's three-eighths rule.**

To show how Simpson's rules may be applied to various magnitudes, we will take the specific case of the approximate determination of the volume of a solid. Let some line of the solid be taken as the X-axis, and let the solid be cut by equidistant planes perpendicular to this axis. Then, if $y_1, y_2, \cdots, y_n$ denote respectively the areas of the plane sections, formula (6) or formula (8) gives approximately the value of the integral $\int y\,dx$, taken between proper limits, and therefore the volume of the solid.

Ex. **1**. Find approximately the area under the equilateral hyperbola $xy = 84$, from $x = 2$ to $x = 8$.

Taking unit intervals, we have for $x = 2, 3, \cdots, 8$, $y = 42, 28, 21, 16.8, 14, 12, 10.5$. By Simpson's one-third rule,

$$A = \tfrac{1}{3}[42 + 10.5 + 2(21 + 14) + 4(28 + 16.8 + 12)] = 116.67.$$

By the three-eighths rule,

$$A = \tfrac{3}{8}[42 + 10.5 + 2 \times 16.8 + 3(28 + 21 + 14 + 12)] = 116.67.$$

The exact area is $\int_2^8 y\,dx = 84\int_2^8 \frac{dx}{x} = 84 \log\frac{8}{2} = 116.45.$

Ex. **2**. Find approximately the value of π from the formula

$$\frac{\pi}{4} = \text{arc tan } 1 = \int_0^1 \frac{dx}{1 + x^2}.$$

Here $y = \frac{1}{1+x^2}$, and dividing the interval (0, 1) into 10 parts, whence $h = 0.1$, we get for the successive ordinates,

$y_1 = 1$	$y_5 = .8620690$	$y_9 = .6097561$
$y_2 = .9900990$	$y_6 = .8000000$	$y_{10} = .5524862$
$y_3 = .9615385$	$y_7 = .7352941$	$y_{11} = .5000000$
$y_4 = .9174312$	$y_8 = .6711409$	

By Simpson's one-third rule, we get

$$\frac{\pi}{4} = .78539815, \text{ whence } \pi = 3.141593.$$

This result is correct to seven figures.

Ex. **3**. Find by Simpson's rule the volume of a sphere of radius a.

Dividing the diameter into four intervals, we have $h = \frac{a}{2}$, and $y = 0$, $y = \frac{3}{4}\pi a^2$, $y = \pi a^2$, $y = \frac{3}{4}\pi a^2$, $y = 0$. Hence by Simpson's first rule,

$$\text{Volume} = \frac{1}{3} \cdot \frac{a}{2}\left[0 + 4\left(\frac{3}{4}\pi a^2 + \frac{3}{4}\pi a^2\right) + 2\pi a^2\right] = \frac{4}{3}\pi a^3.$$

EXERCISES

1. Find $\int_1^{11} x^2\,dx$ by each of Simpson's rules. Take $h = 1$.

2. Find log 2 approximately from the formula $\log 2 = \int_0^1 \frac{dx}{1+x}$. Take 10 intervals and use the one-third rule.

3. Find $\int_0^{\frac{\pi}{3}} \sin\theta\,d\theta$. Take $h = 10° = \frac{\pi}{18}$.

4. Air at a pressure of 40 pounds per square inch expands from a volume of 6 cubic feet to a volume of 20 cubic feet. The expansion follows the law $pv = C$. Find approximately by Simpson's rule the work done during the expansion. Remember that the units must be consistent.

5. Find by Simpson's rule the volume of the frustum of a cone whose base radii are r_1 and r_2 and whose altitude is h.

6. Show that the area under the curve $y = kx^2$ from $x = 0$ to $x = a$ represents the volume of a cone of altitude a.

121. Mechanical integration. Instruments called *mechanical integrators* or *planimeters* have been devised, by means of which plane areas can be measured very accurately. A tracing point is made to follow the perimeter of the figure to be measured, and the area is given by the reading of a recording wheel. For a descrip-

tion of the planimeter and a discussion of its theory the student is referred to the standard works on engineering.*

MISCELLANEOUS EXERCISES

Integrate the following.

1. $\int \frac{dx}{x^2(1+x^2)^{\frac{3}{2}}}$. **2.** $\int \frac{dx}{\sqrt{4+3x-2x^2}}$. **3.** $\int \frac{\sqrt[3]{x}-\sqrt{x}}{4x^{\frac{1}{4}}}\,dx$.

4. $\int \frac{dx}{\sqrt{a+\sqrt{x}}}$. **5.** $\int \frac{x^3\,dx}{\sqrt{2x^2+1}}$. **6.** $\int \frac{\cos\theta\,d\theta}{\cos(\theta+\epsilon)}$.

7. Integrate in three different ways each of the following.

$$(a)\ \int (a^2-x^2)^{\frac{3}{2}}\,dx; \qquad (b)\ \int x^2\sqrt{a^2-x^2}\,dx.$$

8. Show that $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx.$

SUGGESTION. Use the substitution $\tan^2 x = \sec^2 x - 1$.

9. Making use of the reduction formula of Ex. 8, find

$$(a)\ \int \tan^5 x\,dx; \qquad (b)\ \int \tan^4\theta\,d\theta.$$

Perform the following integrations.

10. $\int \frac{3x-2}{x^2-5x+4}\,dx$. **11.** $\int x^2 \operatorname{arc\,cos} x\,dx$.

12. $\int e^{\operatorname{arc\,tan} x}\frac{dx}{(1+x^2)^{\frac{3}{2}}}$. **13.** $\int e^{2x}\cos^2 x\,dx$.

14. Integrate $\int \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}}\,dx$ by the substitution $x^2 = a^2\cos 2\theta$.

15. By the method of integration by parts deduce the following formulas.

$$(a)\ \int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx;$$

$$(b)\ \int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

16. Using the formulas of Ex. 15 find the following integrals.

$$(a)\ \int \sin^5\theta\,d\theta; \qquad (b)\ \int \cos^4\theta\,d\theta.$$

---

* See, for example, Johnson's *Surveying* or Carpenter's *Experimental Engineering*; also the descriptive pamphlets issued by firms manufacturing such instruments.

17. Prove that $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$

$$= \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots n} \cdot \frac{\pi}{2}, \text{ if } n \text{ is an even integer,}$$

$$= \frac{2\cdot 4\cdot 6 \cdots (n-1)}{3\cdot 5\cdot 7 \cdots n}, \text{ if } n \text{ is an odd integer.}$$

Evaluate the following definite integrals.

18. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \theta \sin\theta \cos\theta\, d\theta.$ **19.** $\int_0^{\pi} \theta \sin^2\theta \cos\theta\, d\theta.$ **20.** $\int_0^{\frac{\pi}{2}} \frac{\sin 4\theta}{\sin\theta} d\theta.$

Integrate the following.

21. $\int \frac{dx}{(1-x^2)\sqrt{1+x^2}}.$ **22.** $\int \frac{x\, dx}{(x^2+a^2)(x^2+b^2)}.$ **23.** $\int x^3\sqrt{a+bx^2}\, dx.$

24. By means of the substitution $z = \tan\frac{x}{2}$ verify the following results:

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \arctan\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right], \quad (a^2 > b^2),$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}, \quad (a^2 < b^2).$$

$$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \arctan\left[\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}\right], \quad (a^2 > b^2),$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}}, \quad (a^2 < b^2).$$

25. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

26. Find the area bounded by the curve $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

27. Find the area between the cissoid $y^2 = \frac{x^3}{2a-x}$ and its asymptote $x = 2a$.

28. Find the area under the curve $y = \log x$ from $x = 1$ to $x = 10$.

29. Find the area of the loop of the curve $my^2 = (x-a)(x-b)^2$.

30. Find the length (*a*) of the curve $y = e^x$ from $x = 0$ to $x = 2$; (*b*) of the curve $y = \log x$ from $x = 1$ to $x = 6$; (*c*) of the curve $9y^2 = x(x-3)^2$ from $x = 0$ to $x = 3$.

31. Find the length of the spiral $\rho = a\theta$ from the origin to the point $(\pi a, \pi)$.

32. Find the length of an arc of the cissoid $\rho = 2a\dfrac{\sin^2\theta}{\cos\theta}$ from $\theta = 0$ to $\theta = \dfrac{1}{4}\pi$.

33. Find the length of the curve $\rho = a\sin^3\dfrac{\theta}{3}$.

34. A circle of radius $\dfrac{a}{2}$ rolls on a circle of radius a, and a point on the circumference of the rolling circle traces an epicycloid whose polar equation is $4(\rho^2 - a^2)^3 = 27\,a^4\rho^2\sin^2\theta$. Find the whole length of the curve thus traced.

35. Find the volume generated by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the X-axis.

36. Find the volume generated by revolving one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

about the axis OX.

37. Find the volume generated if the cycloid is revolved about OY.

38. Find the volume generated by revolving one arch of the curve $y = \cos x$ about the axis OX.

39. Find the volume generated by revolving the cardioid $\rho = 2a(1 - \cos\theta)$ about the axis OX.

40. Two cylinders, with the same altitude h, have a common upper base of radius a, and the lower bases are tangent to each other. Find the volume common to the two cylinders.

41. Find the surface generated by revolving the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about its major axis; also about its minor axis.

42. Find the surface generated by revolving the cardioid $\rho = 2a(1 - \cos\theta)$ about the initial line.

43. Find the mean length of the ordinates of a semicircle of radius a, if the ordinates are taken at equidistant intervals on the diameter.

44. Find the mean distance of the points on the circumference of a circle of radius a from a fixed point on the circumference.

45. Zeuner's equation for superheated steam is $pv = BT + Cp^n$. For an isothermal expansion the temperature T is constant. Derive an expression for the work done during an isothermal expansion from v_1 to v_2.

SUGGESTION. Use the equation

$$\text{work} = \int p\,dv = pv - \int v\,dp.$$

46. The acceleration of a particle that moves under the influence of an attractive force that varies inversely as the square of the distance is $a = -\dfrac{k}{s^2}$. Derive a relation between v and s, also between t and s, taking s_0 as the initial distance of the particle from the center of attraction.

CHAPTER XIII

FUNCTIONS OF TWO OR MORE VARIABLES

122. Definition of a function of several variables. Heretofore we have discussed functions of a single independent variable. A function, however, may depend upon two or more variables having no mutual relation, that is, independent of one another. Thus the volume of a gas depends upon the temperature and also upon the pressure to which it is subjected, and the pressure and temperature may vary independently. In general, we may say:

z is a function of the independent variables x, y, $\cdots$ when for each set of values of these variables there is determined a definite value or values of z.

A function of two variables

$$z = f(x, y)$$

is represented geometrically by a surface, and to each pair of values of (x, y) there corresponds a point on this surface. The motion of a point on this surface depends upon the manner in which x and y vary. One of these variables may remain constant while the other is allowed to vary, or the two variables may change simultaneously. In the first case, the extremity of the ordinate describes a plane curve lying in a plane parallel to the YZ- or XZ-plane; and in the second case the extremity of the ordinate may describe a space curve.

When one of the variables remains constant, say $y = y_0$, we say that the function $f(x, y)$ is **continuous in x** at the point (x_0, y_0) if

$$\underset{x \doteq x_0}{L} f(x, y_0) = f(x_0, y_0). \tag{1}$$

Likewise if we have

$$\underset{y \doteq y_0}{L} f(x_0, y) = f(x_0, y_0), \tag{2}$$

we say that $f(x, y)$ is **continuous in y** at the point (x_0, y_0). In order

that the function be **continuous in both variables together** at the point in question we must have

$$\underset{\substack{x \doteq x_0 \\ y \doteq y_0}}{L} f(x, y) = f(x_0, y_0). \tag{3}$$

As will be seen from these limits, the continuity in (x, y) together involves of necessity continuity in x and in y. In general, whenever we speak of the continuity of a function of two variables, the continuity with respect to both variables taken together is to be understood unless otherwise stated.

123. Partial derivatives. In the preceding article it was pointed out that a function of two variables may vary either by permitting one of the variables to remain constant while the other changes, or by allowing both to vary simultaneously. The increment of the function $f(x, y)$ due to a change in x alone is

$$f(x + \Delta x, y) - f(x, y).$$

Let us consider the ratio of this increment to the increment of the variable x. The limit of this ratio as $\Delta x \doteq 0$, viz.:

$$\underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

is called the **partial derivative** of $f(x, y)$ with respect to x. Similarly, the limit

$$\underset{\Delta y \doteq 0}{L} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

is called the partial derivative of $f(x, y)$ with respect to y. These derivatives are called partial derivatives because they measure only partially the variation of the function as compared with that of the variables. To distinguish these from the derivatives which have been thus far considered, we shall call the latter **total derivatives.** In a subsequent article we shall discuss methods of determining total derivatives of functions of several variables.

To distinguish symbolically the two classes of derivatives, we denote the partial derivatives by using the round ∂ instead of d. Thus the partial derivatives of $z = f(x, y)$ with respect to x and y are written

$$\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y},$$

respectively. They are frequently represented also by the equivalent symbols

$$f_x'(x, y),\ f_y'(x, y).$$

Partial derivatives usually involve both x and y, and may likewise have partial derivatives with respect to either variable. Thus we may have

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),\ \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

These higher partial derivatives are represented symbolically by

$$\frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial y^2},\ \frac{\partial^2 z}{\partial x\,\partial y},\ \frac{\partial^2 z}{\partial y\,\partial x},$$

or by

$$f_x''(x, y),\ f_y''(x, y),\ f_{xy}''(x, y),\ f_{yx}''(x, y),$$

respectively.

We may extend these considerations to partial derivatives of still higher order and to functions of more than two variables. The notation employed for the partial derivatives of higher order indicates the number of differentiations and the order in which they are made. Thus, $\frac{\partial^3 u}{\partial x\,\partial y^2}$ indicates three differentiations, the first two with respect to y, the third with respect to x; $\frac{\partial^5 u}{\partial z\,\partial x^2\,\partial y^2}$ indicates five differentiations, the first and second with respect to y, the third and fourth with respect to x, and the fifth with respect to z.

Since the function $z=f(x, y)$ is represented by a surface, $z=f(x, b)$ is the equation of a plane curve cut from this surface by the plane $y=b$. For the derivative $\frac{\partial z}{\partial x}$ we have the same geometric interpretation in this plane as was given earlier for $\frac{dz}{dx}$, namely, the slope of the tangent to the curve in question.

Let $z=f(x, y)$ be the surface shown in Fig. 70. Consider any point on this surface, as P, at the intersection of the curves BPC and EPF, cut from the surface by the planes $y=b$ and $x=a$, respectively. Then the slope of the curve BPC is given by the

partial derivative $\frac{\partial z}{\partial x}$, and that of the curve EPF by the partial derivative $\frac{\partial z}{\partial y}$. That is,

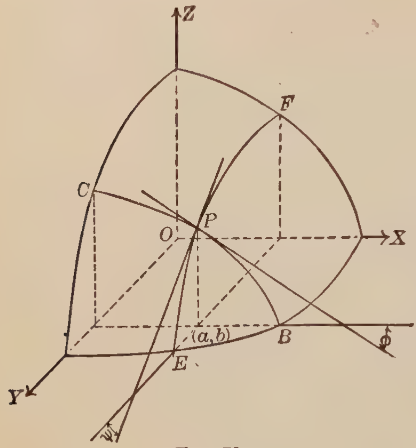

FIG. 70.

$$\tan\phi = \frac{\partial z}{\partial x},\quad \tan\psi = \frac{\partial z}{\partial y}.$$

The values of $\tan\phi$ and $\tan\psi$ for some definite point P on the surface are obtained by substituting in the expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, respectively, the corresponding values of x and y. Thus if (a, b) is the projection of P on the XY-plane, we substitute a for x and b for y.

The process of finding a partial derivative is in all respects the same as that employed in finding an ordinary derivative of a function of a single variable. The following examples illustrate this statement.

Ex. **1.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, when $z = y^2 \sin x$.

Treating y as a constant and differentiating with respect to x, we have

$$\frac{\partial z}{\partial x} = y^2 \cos x.$$

Likewise, if we consider x constant and differentiate with respect to y, we get

$$\frac{\partial z}{\partial y} = 2\,y \sin x.$$

Ex. **2.** Find successive partial derivatives of the function $xe^x \log y$.

We have
$$z = xe^x \log y.$$

$$\frac{\partial z}{\partial x} = (1+x)\,e^x \log y\,;\quad \frac{\partial z}{\partial y} = \frac{xe^x}{y}.$$

$$\frac{\partial^2 z}{\partial x^2} = (2+x)\,e^x \log y\,;\quad \frac{\partial^2 z}{\partial y^2} = -\frac{xe^x}{y^2}.$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{(1+x)\,e^x}{y};\qquad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{(1+x)\,e^x}{y}.$$

EX. 3. The surface $z = \frac{x^2}{8} + \frac{y^2}{12}$ is cut by planes $x = 4$, $y = 3$. Find the slopes of the curves cut from the surface by these planes at the point of intersection of the curves.

We have here

$$\frac{\partial z}{\partial x} = \frac{x}{4}, \quad \frac{\partial z}{\partial y} = \frac{y}{6}.$$

At the specified point, therefore,

$$\tan \phi = \frac{x}{4}\Big]_{x=4} = 1; \ \tan \psi = \frac{y}{6}\Big]_{y=3} = \frac{1}{2}.$$

EXERCISES

Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following functions.

1. $z = x^2y^5$.

2. $z = \sin x \cos y$.

3. $z = ye^x + xe^y$.

4. $z = x^4 - ax^2y + bxy^2 + y^4$.

5. $z = y^x$.

6. $z = \text{arc sec } \frac{y}{x}$.

7. $z = y \sin x + x \sin y$.

8. $z = \sin (x + y)$.

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ for the following functions of three variables.

9. $u = 3\,x^2yz^{-\frac{3}{2}}$.

10. $u = e^x \log yz$.

11. $u = \sin x \cos y + \sin y \cos z + \sin z \cos x$.

12. $u = \log \frac{x+y}{z}$.

13. $u = \frac{x^3 + y^3 + z^3}{xyz}$.

14. The volume of a cone may be expressed as a function of the altitude and of the radius of the base. Denoting the volume by V, the base radius by r, and the altitude by h, express V in terms of r and h, find the partial derivatives $\frac{\partial V}{\partial r}$, $\frac{\partial V}{\partial h}$, and give interpretations of these derivatives.

15. Express the area A of a triangle as a function of its base x and altitude y. Find the rate at which the area changes: (a) when the altitude remains unchanged and the base varies; (b) when the base is constant and the altitude varies.

16. Express the area A of a triangle in terms of two sides m and n and the included angle θ. Find the rate of change of A: (a) when θ changes, m and n remaining the same; (b) when m changes, θ and n remaining the same.

17. Interpret geometrically the equations

$$\frac{\partial(xy)}{\partial x} = y, \quad \frac{\partial(xy)}{\partial y} = x.$$

Form $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\, \partial y}$, $\frac{\partial^2 u}{\partial y\, \partial x}$, and $\frac{\partial^2 u}{\partial y^2}$ for the following functions.

18. $u = x \log y$. **19.** $u = x^m y^n$. **20.** $u = x^3 + axy - y^3$.

21. $u = e^{x+y}$. **22.** $u = x(1 - y^3)$.

23. The formula $H = ks^3 D^{\frac{2}{3}}$ is used to determine the horse power required to drive a steamship, where s denotes the speed and D the displacement. What is the interpretation of the derivative $\frac{\partial H}{\partial s}$? of $\frac{\partial H}{\partial D}$? Derive expressions for these derivatives.

124. Interchange of order of differentiation. Among the partial derivatives enumerated in Art. 123 were the second derivatives $\frac{\partial^2 f}{\partial x\, \partial y}$, $\frac{\partial^2 f}{\partial y\, \partial x}$. In most cases that arise in the applications of the calculus to physical problems, it is a matter of indifference in what order the differentiation is performed; that is, in most cases these two partial derivatives give the same result.

Let us consider the limits involved. We have, by definition,

$$\frac{\partial f}{\partial x} = \underset{\Delta x \doteq 0}{L} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{1}$$

$$\frac{\partial f}{\partial y} = \underset{\Delta y \doteq 0}{L} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{2}$$

Consequently, we have

$$\frac{\partial^2 f}{\partial y\, \partial x} = \underset{\Delta y \doteq 0}{L} \; \underset{\Delta x \doteq 0}{L} \frac{\dfrac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x} - \dfrac{f(x+\Delta x, y) - f(x, y)}{\Delta x}}{\Delta y}$$

$$= \underset{\Delta y \doteq 0}{L} \; \underset{\Delta x \doteq 0}{L} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y\, \Delta x}; \tag{3}$$

and similarly,

$$\frac{\partial^2 f}{\partial x\, \partial y} = \underset{\Delta x \doteq 0}{L} \; \underset{\Delta y \doteq 0}{L} \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x, y)}{\Delta y\, \Delta x}. \tag{4}$$

It appears that the only difference between the two derivatives is the order in which the increments Δx and Δy are allowed to approach zero. It can be shown that this order is always a matter of indifference if f_x', f_{yx}'' (or f_y', f_{xy}'') are continuous functions of the two variables (x, y) taken together.*

EXERCISES

In each of the following functions show that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$.

1. $u = x^3y^2 - 4\,xy^4$. **2.** $u = \cos(x + y)$. **3.** $u = e^x \sin y$.

4. $u = y^x$. **5.** $u = \log(x^2 + y^2)$. **6.** $u = \cos xy^2$.

For each of the following functions show that $\dfrac{\partial^3 u}{\partial x\,\partial y^2} = \dfrac{\partial^3 u}{\partial y^2\,\partial x}$.

7. $u = x^2(x - y)$. **8.** $u = xy\cos(x + y)$.

9. $u = y\log(1 + xy)$. **10.** $u = \sin^2 x \cos y$.

125. Total derivatives. Let $z = f(x, y)$ be a continuous function of the two independent variables, having continuous derivatives; and let both x and y be arbitrarily chosen continuous functions of a common variable t, having the continuous derivatives $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$. We shall attempt to find an expression for $\dfrac{dz}{dt}$ in terms of $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$, in other words, to express the rate of change of z in terms of the rates of change of x and y.

For the increment of z, we have

$$\Delta z = f(x + \Delta x,\, y + \Delta y) - f(x, y),$$

which may be written in the form

$$\Delta z = [f(x+\Delta x,\, y+\Delta y) - f(x,\, y+\Delta y)] + [f(x,\, y+\Delta y) - f(x, y)]. \quad (1)$$

By an application of the law of the mean, we have

$$f(x + \Delta x,\, y + \Delta y) - f(x,\, y + \Delta y) = f_x'(x + \theta_1 \Delta x,\, y + \Delta y)\,\Delta x, \quad (2)$$

$$f(x,\, y + \Delta y) - f(x, y) = f_y'(x,\, y + \theta_2 \Delta y)\,\Delta y, \quad (3)$$

* See *First Course*, pp. 247 et seq.

where θ_1 and θ_2 lie between 0 and 1. Substituting these values in the second member of (1), we have, after dividing by Δt,

$$\frac{\Delta z}{\Delta t} = f_x'(x + \theta_1 \Delta x, y + \Delta y)\frac{\Delta x}{\Delta t} + f_y'(x, y + \theta_2 \Delta y)\frac{\Delta y}{\Delta t}. \qquad (4)$$

In the limit, we have

$$\underset{\Delta t \doteq 0}{L} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}, \quad \underset{\Delta t \doteq 0}{L} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \quad \underset{\Delta t \doteq 0}{L} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}. \qquad (5)$$

Moreover, since Δx and Δy approach zero with Δt, and $f_x'(x, y)$ and $f_y'(x, y)$ are continuous functions of both variables together, we have

$$\underset{\Delta t \doteq 0}{L} f_x'(x + \theta_1 \Delta x, y + \Delta y) = f_x'(x, y), \qquad (6)$$

$$\underset{\Delta t \doteq 0}{L} f_y'(x, y + \theta_2 \Delta y) = f_y'(x, y). \qquad (7)$$

Writing $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for $f_x'(x, y), f_y'(x, y)$, respectively, we have from (4)

$$\boldsymbol{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}. \qquad (a)$$

This result may be expressed in words as follows:

The total rate of change of a function of x and y is made up of two parts: one is the rate of change of x multiplied by the partial x-derivative of the function, the other the rate of change of y multiplied by the partial y-derivative.

The following examples will serve to illustrate this principle.

EX. **1.** Suppose the characteristic equation of a gas to be $pv = 54\,T$, and let the volume and temperature at a given instant be $v_0 = 15$ and $T_0 = 640$. The corresponding pressure is

$$p_0 = \frac{54 \times 640}{15} = 2304.$$

In this state, suppose the temperature to be rising at the rate of 0.5 degree per minute and the volume to be increasing at the rate of 0.2 cubic foot per minute. Required the rate at which the pressure is changing.

We have $$p = 54\frac{T}{v},$$

whence $$\frac{\partial p}{\partial T} = \frac{54}{v}, \quad \frac{\partial p}{\partial v} = -54\frac{T}{v^2}$$

Hence, in the given state, $\left.\frac{\partial p}{\partial T}\right]_{T=T_0} = \frac{54}{15} = 3.6,$

and $\left.\frac{\partial p}{\partial v}\right]_{v=v_0} = -\frac{54 \times 640}{15^2} = -153.6.$

Also, taking the time t as the auxiliary variable,

$$\frac{dT}{dt} = 0.5, \text{ and } \frac{dv}{dt} = 0.2.$$

Then, $\frac{dp}{dt} = \frac{\partial p}{\partial T}\frac{dT}{dt} + \frac{\partial p}{\partial v}\frac{dv}{dt} = 3.6 \times 0.5 - 153.6 \times 0.2 = -28.92.$

That is, the pressure is decreasing at the rate of 28.92 lb. per square foot per minute.

Ex. **2**. The equation $z = 2x^2 + 5y^2$ represents an elliptic paraboloid. At the point $x = -3$, $y = 1$, we have

$$\left.\frac{\partial z}{\partial x}\right]_{x=-3} = 4x\Big]_{x=-3} = -12,$$

$$\left.\frac{\partial z}{\partial y}\right]_{y=1} = 10y\Big]_{y=1} = 10.$$

Assume the rate of change of x to be 3 units per second, and that of y to be 2 units per second; that is, $\frac{dx}{dt} = 3$ and $\frac{dy}{dt} = 2$. Then the rate of change of z is

$$\frac{dz}{dt} = -12\frac{dx}{dt} + 10\frac{dy}{dt} = -16 \text{ units per second.}$$

Let us now suppose that the variables x and y are not independent, but that they are connected by some relation that may be expressed explicitly by the equation

$$y = F(x),$$

or implicitly by the relation

$$\phi(x, y) = 0.$$

This relation restricts the moving point (x, y) to a particular curve in the XY-plane and consequently the values of z to a particular curve in space, that is, to the intersection of the cylinder $\phi(x, y) = 0$ and the surface $z = f(x, y)$. Whenever such a relation as the above exists between x and y, we may choose x itself as the variable t, and formula (a) reduces to the following:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \qquad (b)$$

In this formula it is to be observed that $\frac{dz}{dx}$ and $\frac{\partial z}{\partial x}$ have very different meanings. In finding the partial derivative $\frac{\partial z}{\partial x}$ it is assumed that x alone varies, that is, that y is constant. On the other hand, $\frac{dz}{dx}$ expresses the variation of z due to a change in both x and y; for this reason it is called a **total derivative.** Likewise in (a), $\frac{dz}{dt}$ is the total derivative of z with respect to t. Moreover, the value of $\frac{\partial z}{\partial x}$ is definitely determined by the value of the coördinates of the point in question; in other words, it is a function of the coördinates only. This follows from the fact that the variation can take place in one direction only. Functions like this, which depend for their values solely upon the coördinates of the point, are called **point functions.** The value of $\frac{dz}{dx}$ depends, however, not only upon the coördinates of the point, but also upon the direction in which that point is approached. This derivative is therefore not a point function.

Formulas (a) and (b) may be extended to functions of any number of variables. Thus, if

$$u = f(x, y, z),$$

we have
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}; \qquad (c)$$

and if further $y = \phi(x)$, $z = \psi(x)$,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}. \qquad (d)$$

The proof is left as an exercise for the student.

Formulas (b) and (d) are useful in the differentiation of somewhat complicated functions of a single variable. The following example shows such an application.

Ex. 3. Find $\frac{du}{dx}$, where $u = xe^{\sqrt{a^2-x^2}}\sin^3 x$.

Let $y = \sqrt{a^2 - x^2}$ and $z = \sin^3 x$; then $u = xe^y z$,

and
$$\frac{\partial u}{\partial x} = e^y z, \quad \frac{\partial u}{\partial y} = xe^y z, \quad \frac{\partial u}{\partial z} = xe^y.$$

Also $$\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}, \quad \frac{dz}{dx} = 3\sin^2 x \cos x.$$

Substituting these expressions for the derivatives in (d), we get

$$\frac{du}{dx} = e^y z - \frac{x^2 e^y z}{\sqrt{a^2 - x^2}} + 3\, x e^y \sin^2 x \cos x$$

$$= e^{\sqrt{a^2 - x^2}} \sin^3 x \left[1 - \frac{x^2}{\sqrt{a^2 - x^2}} + 3 \cot x\right].$$

EXERCISES

Find $\frac{du}{dx}$ in each of the following.

1. $u = x^2 + y^2$, and $y = e^{\sin x}$.

2. $u = \text{arc} \tan \frac{y}{x}$, and $y = e^x$.

3. $u = \log(x + y)$, and $y = \sqrt{x^2 + a^2}$.

4. $u = x^2 e^{\frac{1}{x}} \cos x$.

5. $u = e^{ax}(y - z)$, and $y = a \sin x$, $z = \cos x$.

6. A triangle has a base of 10 units and an altitude of 6 units. The base is made to increase at the rate of 2 units, and the altitude to decrease at the rate of $\frac{1}{2}$ unit. At what rate does the area change?

7. A point lying on the ellipsoid $\frac{x^2}{36} + \frac{y^2}{25} + \frac{z^2}{49} = 1$ in the position $x = 3$, $y = -4$, moves so that x increases at the rate of two units per second, while y decreases at the rate of three units per second. Find the rate of change of z.

8. With the same data as in illustrative Ex. 1, suppose the pressure of the gas to be increasing at the rate of 40 pounds per square foot per second, while the temperature is falling at the rate of 1 degree per second. Find the rate of change of the volume.

9. A gas has the equation $pv = RT$, and expands following the law $pv^n = C$, where n and C are constants. Derive expressions for the total derivatives $\frac{dT}{dv}$ and $\frac{dT}{dp}$ under these conditions.

10. Derive the same results by eliminating p and v successively between the characteristic equation of the gas and the equation of the expansion and differentiating the resulting expressions.

126. Total differentials. In formula (a), $\frac{dz}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ are the quotients of two differentials; hence we may multiply through by the differential dt. The resulting formula

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \tag{e}$$

expresses the differential of z in terms of the differentials of x and y. This formula may be extended to any number of variables. Thus if

$$u = f(x_1, x_2, x_3, \cdots, x_n),$$

$$du = \frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \frac{\partial u}{\partial x_3}\,dx_3 + \cdots + \frac{\partial u}{\partial x_n}\,dx_n.$$

The differential dz in the formula (e) is called the **total differential** of z. The terms $\frac{\partial z}{\partial x}dx$ and $\frac{\partial z}{\partial y}dy$ are called respectively the **partial x-differential** and **partial y-differential** of z. These latter are frequently denoted by $d_x z$ and $d_y z$, and (e) is then written in the form

$$dz = d_x z + d_y z. \tag{e'}$$

Equation (e) therefore expresses symbolically the principle that *the total differential of a function of several variables is equal to the sum of the partial differentials.*

Ex. 1. Let $u = e^x y^2 \sin z$. Then

$$\frac{\partial u}{\partial x} = e^x y^2 \sin z, \quad \frac{\partial u}{\partial y} = 2\,e^x y \sin z, \quad \frac{\partial u}{\partial z} = e^x y^2 \cos z.$$

Therefore $\quad du = e^x y^2 \sin z\,dx + 2\,e^x y \sin z\,dy + e^x y^2 \cos z\,dz.$

The differential dz of a function of two variables is susceptible of a geometrical representation similar to that for the differential of a function of a single variable (Art. 46). Let $PRQS$, Fig. 71, be an element of the surface $z = f(x, y)$, and let a tangent plane $PEGF$ be passed through the point P, whose coördinates are (x_1, y_1, z_1). The arbitrary increments of the independent variables x and y are $\Delta x = dx$ and $\Delta y = dy$, respectively. From the figure we have evidently

$$BD = AE = PA \tan EPA = \frac{\partial z}{\partial x}dx, \text{ and } DG = CF = \frac{\partial z}{\partial y}dy.$$

Hence $\quad BG = BD + DG = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy,$

that is, $\quad BG = dz.$

The increment of z corresponding to the increments Δx, Δy of x and y is $\Delta z = BQ$; hence, the total differential dz is usually different from the increment Δz. The two symbols dz and Δz represent the same value when the surface given by $z = f(x, y)$ is a plane.

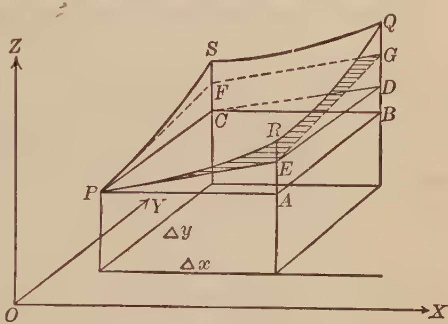

FIG. 71.

It is worthy of notice that since dz as given by formula (e) is an approximation to Δz, we may use that formula to calculate approximately the effect on a function z of small errors Δx and Δy in the observed or measured values of the variables x and y.

Ex. Given $z = x^2 - xy$, find approximately the increment of z corresponding to the assumed increments $\Delta x = 0.02$, $\Delta y = 0.01$, when $x = 8$, $y = 5$.

From formula (e), $dz = (2x - y)\,\Delta x - x\,\Delta y = 0.22 - 0.08 = 0.14$. The actual change Δz is found to be 0.1402.

EXERCISES

Differentiate each of the following functions by means of formula (e) and verify the result by direct differentiation.

1. $z = e^x y^2$.

2. $z = y^x$.

3. $z = \sin x \cos y$.

4. $z = \dfrac{x^2}{y^3}$.

5. $z = a^x e^y$.

6. $z = \arctan \dfrac{x}{y}$.

7. $z = x^3 y^{\frac{1}{2}} - x^{-\frac{3}{2}} y^2$.

8. $z = \arccos \dfrac{y}{x} + \arctan \dfrac{x}{y}$.

Differentiate each of the following functions.

9. $u = x^3 + y^3 - 2xyz$.

10. $u = z^{xy}$.

11. $u = \sin x \cos y \tan z$.

12. $u = \arctan \dfrac{xy}{z}$.

13. $u = \dfrac{x^2 z}{a^3 - y^3}$.

14. From the perfect gas equation $pv = RT$, find the change in p produced by simultaneous changes of v and T.

15. The formula for the coefficient of diffusion of a gas is $k = CpT^n$, where C is a constant, p the pressure, and T the absolute temperature. Find the change in the coefficient due to simultaneous changes in pressure and temperature.

16. Taking the formula $H = ks^3D^{\frac{2}{3}}$ for the horse power of a steamship, derive an approximate expression for the increase in horse power due to an increase Δs in the speed, and an increase ΔD in the displacement.

17. If $u = x^2y$, find approximately the change in u when x changes from 10 to 10.02 and y from 4 to 4.01. Calculate also the change in u when $\Delta x = 0.002$ and $\Delta y = 0.001$. In each case compare these approximate changes in u with the actual changes as derived from the original equation. Interpret the results in connection with Fig. 71.

18. Let $z = \sin x \cos y$. If $x = 22°$, $y = 37°$, find the change of z due to the changes $\Delta x = 10'$, $\Delta y = 15'$. By formula (e), calculate the approximate change and compare the result with the actual change.

19. Let b and h be respectively the breadth and height of a rectangle and A the area; then $A = bh$. Interpret geometrically the approximate equation

$$\Delta A = h\,\Delta b + b\,\Delta h,$$

and show wherein the approximation lies.

20. Find the relative error in the computed area of an ellipse due to errors in the measurement of its semi-axes a and b.

21. Given a triangle having sides a, b, c, and opposite angles A, B, and C. If a side c is determined by measuring two sides a and b and the included angle C, show that the error Δc is given approximately by the equation

$$\Delta c \doteq \Delta a \cos B + \Delta b \cos A + a\,\Delta C \sin B.$$

127. Differentiation of implicit functions. Suppose that y is defined implicitly as a function of x by means of the relation $f(x, y) = 0$. Since the function $f(x, y)$ has by definition the constant value zero for all values of x and y, the total differential must also be zero. Hence, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0. \tag{1}$$

Transposing the first term to the second member of the equation and dividing both members by dx, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \tag{f}$$

which may be used as a formula for writing out at once the derivative $\frac{dy}{dx}$ of an implicit function. This method is easier to apply than the earlier one given in Art. 49, and should be used in practice.

Ex. Given $f(x, y) = x^2y - xy^3 = 0$.

We have $$\frac{\partial f}{\partial x} = 2\,xy - y^3,\quad \frac{\partial f}{\partial y} = x^2 - 3\,xy^2;$$

therefore, $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2\,xy - y^3}{x^2 - 3\,xy^2}.$$

EXERCISES

For each of the following functions find the first derivative by the method of this article.

1. $y^3 - 3\,x^2y + xy^2 = 0$.

2. $x^3 \sin y - y^3 \cos x = 0$.

3. $xy^n = C$.

4. $(x^2 + y^2)^2 + y^2(x - 2\,a) = 0$.

5. $x^4 - 3\,xy^2 + 2\,y^3 = 0$.

6. $e^x \sin y = C$.

7. $p(v - b)^n = C$, find $\frac{dp}{dv}$.

8. $\rho^3 - \rho^2 \cos\theta = C$, find $\frac{d\rho}{d\theta}$.

9. $b^2x^2 + a^2y^2 = a^2b^2$.

128. Exact and inexact differentials. In deriving formulas (a) and (e) for the total derivative and total differential, respectively, we started with a given function $z = f(x, y)$ of two independent variables x and y. In the application of calculus to problems in physics and mechanics, we frequently meet with expressions having forms precisely similar to (a) or (e) yet derived by a quite different process. To illustrate this statement let us consider a physical problem, that of heating a gas. The state of a gas is defined by the absolute temperature T and the volume v, and a change in either T or v is accompanied by the absorption of heat. If now the volume v is kept constant, it is known from experiment that a change of temperature ΔT is accompanied by the absorption of heat $\Delta_T Q = c_m \Delta T$, where c_m denotes the mean specific heat at constant volume for the interval ΔT. If we

assume that T is a function of the time, say $T=\phi(t)$, the time rate of absorption of heat is

$$\underset{\Delta t \doteq 0}{L} \frac{\Delta_T Q}{\Delta t} = \underset{\Delta t \doteq 0}{L}\, c_m \frac{\Delta T}{\Delta t} = \underset{\Delta t \doteq 0}{L}\, c_m \cdot \underset{\Delta t \doteq 0}{L} \frac{\Delta T}{\Delta t},$$

that is,

$$\frac{d_T Q}{dt} = c\frac{dT}{dt}, \tag{1}$$

where c denotes the instantaneous specific heat at the beginning of the interval Δt. But the specific heat c is the rate of absorption of heat with respect to the temperature when v is constant; that is, $c=\frac{\partial Q}{\partial T}$. (See Art. 19.) Hence we may write

$$\frac{d_T Q}{dt} = \frac{\partial Q}{\partial T}\frac{dT}{dt}. \tag{2}$$

A similar course of reasoning leads to the result

$$\frac{d_v Q}{dt} = \frac{\partial Q}{\partial v}\frac{dv}{dt}. \tag{3}$$

If T and v change simultaneously, the total rate of absorption of heat is therefore

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial T}\frac{dT}{dt} + \frac{\partial Q}{\partial v}\frac{dv}{dt}. \tag{4}$$

Multiplying through by dt, we obtain

$$dQ = \frac{\partial Q}{\partial T}dT + \frac{\partial Q}{\partial v}dv. \tag{5}$$

Finally, replacing $\frac{\partial Q}{\partial T}$ by c and $\frac{\partial Q}{\partial v}$ by l, the so-called *latent heat of expansion* at constant temperature, we have

$$dQ = c\,dT + l\,dv. \tag{6}$$

It will be noted that (4) and (5) are in form similar to equations (a) and (e) but that they are derived from observed relations between the increments ΔT, Δv, $\Delta_T Q$, $\Delta_v Q$ and the coefficients l and c, and not by the differentiation of a previously existing functional relation between Q, T, and v.

As a second example, consider the work W of moving a particle in a plane, say the XY-plane. It is shown in mechanics that

$$dW = X\,dx + Y\,dy, \tag{7}$$

where X and Y denote respectively the X- and Y-components of the force acting on the particle. Therefore (see Ex. 15, p. 67), $X = \frac{\partial W}{\partial x}$ and $Y = \frac{\partial W}{\partial y}$, and (7) takes the form

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy. \tag{8}$$

Again (8) is not derived by differentiating a function $W = f(x, y)$; it is deduced from the laws of mechanics and the question of a functional relation between W and the coördinates x and y does not enter into consideration in this deduction.

When we have given an expression like $c\,dT + l\,dv$ or $X\,dx + Y\,dy$ the question arises: Can this expression be produced by the differentiation of some function of the variables involved; for example, can we find any function as $Q = \phi(T, v)$ that upon differentiation will produce (6) or any function as $W = \psi(x, y)$ that will likewise produce (7)? To state the question more generally, if M and N are any arbitrarily chosen functions of x and y, does a function of the independent variables (x, y) exist that will upon differentiation produce $M\,dx + N\,dy$? Slight consideration will show that such a function usually does not exist, and if it does exist, the coefficients M and N must satisfy a certain condition. Let it be assumed that there is such a function, say $z = \phi(x, y)$. Then by differentiation we obtain

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \tag{9}$$

If therefore the differentiation of the given function produces $M\,dx + N\,dy$, we must have

$$M = \frac{\partial z}{\partial x},\ N = \frac{\partial z}{\partial y}; \tag{10}$$

that is, M and N must be the partial derivatives of the function z with respect to x and y, respectively. From Art. 124, we have, with proper restrictions regarding continuity,

$$\frac{\partial^2 z}{\partial y\,dx} = \frac{\partial^2 z}{\partial x\,dy} \quad \text{or} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right).$$

Hence, if $M = \frac{\partial z}{\partial x}$ and $N = \frac{\partial z}{\partial y}$, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{11}$$

as the necessary condition that $M\,dx + N\,dy$ may be produced by the differentiation of a function $\phi(x, y)$. It may also be shown that this condition is sufficient.*

If the condition (11) is satisfied, $M\,dx + N\,dy$ is called an **exact differential**; if the condition is not satisfied, $M\,dx + N\,dy$ is called an **inexact differential.**

Ex. **1.** Given $M\,dx + N\,dy = \frac{y}{x}\,dx + \log x\,dy$.

Here $M = \frac{y}{x}$, $N = \log x$, $\frac{\partial M}{\partial y} = \frac{1}{x}$, $\frac{\partial N}{\partial x} = \frac{1}{x}$. The condition imposed by (11) is satisfied and the differential is exact. It is easily seen that the function is $y \log x$.

Ex. 2. Given $M\,dx + N\,dy = x^2y\,dx - 2\,xy\,dx$.

In this case we have $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2y) = x^2$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-2\,xy) = -2\,y$. The given differential is therefore inexact, and no function of x and y exists, the differentiation of which will produce this differential.

The essential difference between exact and inexact differentials appears more clearly when integration is attempted. If the differential $M\,dx + N\,dy$ is exact, it is then the total differential of some function $z = \phi(x, y)$ of the independent variables x and y. We have therefore

$$\int (M\,dx + N\,dy) = \int dz = \phi(x, y) + C. \tag{12}$$

If we assign limits of integration, as (x_1, y_1) and (x_2, y_2), we have

$$\int_{x_1, y_1}^{x_2, y_2} (M\,dx + N\,dy) = \phi(x_2, y_2) - \phi(x_1, y_1). \tag{13}$$

The value of the integral therefore depends only upon the initial and final values of the variables. If we represent (x_1, y_1) and (x_2, y_2) by points in the XY-plane, we say that the integral depends on

---

* See *First Course*, Art. 160.

the end points only and not at all upon the path by which the variable point moves from one to the other. The integral is thus a *point function* of the coördinates x, y.

Ex. 3. Given the differential $y\,dx + x\,dy$.

This differential satisfies the condition (11) and is therefore exact; and since by inspection $y\,dx + x\,dy = d(xy)$, we have

$$\int_{x_1, y_1}^{x_2, y_2}(y\,dx + x\,dy) = \int_{x_1, y_1}^{x_2, y_2} d(xy) = x_2y_2 - x_1y_1.$$

This integral is represented geometrically by the shaded area, Fig. 72, and is evidently independent of the path p between the point (x_1, y_1) and (x_2, y_2).

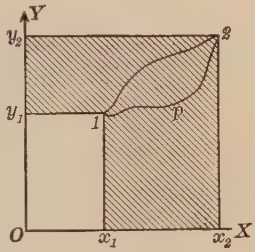

FIG. 72.

The integration of an exact differential may be effected by the following rule, which is sufficient for most cases that arise in practice.

Integrate $M\,dx$ considering y as a constant, then integrate the terms in $N\,dy$ that do not contain x, and take the sum of the two integrals.

Ex. 4. Given $dz = (3\,x^2 + 2\,y^2)dx + (4\,xy - 9\,y^2)dy$.

Since $\dfrac{\partial}{\partial y}(3\,x^2 + 2\,y^2) = \dfrac{\partial}{\partial x}(4\,xy - 9\,y^2) = 4\,y$, the differential is exact.

Integrating $M\,dx$ with y constant, we have

$$\int M\,dx = \int (3\,x^2 + 2\,y^2)dx = x^3 + 2\,xy^2.$$

The part of N that does not contain x is $-9\,y^2$, and $\int -9\,y^2\,dy = -3\,y^3$. Therefore the integral is $z = x^3 + 2\,xy^2 - 3\,y^3 + C$.

If $Mdx + Ndy$ is an inexact differential, no function $\phi(x, y)$ can be found the differentiation of which will produce this differential. Consequently the integral $\int_{x_1, y_1}^{x_2, y_2}(M\,dx + N\,dy)$ cannot be expressed as the difference $\phi(x_2, y_2) - \phi(x_1, y_1)$, and to arrive at any definite result we must assume a relation between x and y, as $y = F(x)$. From this relation we can obtain an expression for the derivative $\dfrac{dy}{dx}$; then by means of the identity,

$$M\,dx + N\,dy = \left(M + N\frac{dy}{dx}\right)dx,$$

we can express the integrand in terms of x and integrate in the usual manner. The following example illustrates this process.

Ex. 5. Investigate the integral, $\int_{0,\,0}^{1,\,2}(y^2\,dx - x\,dy)$.

Since $\dfrac{\partial M}{\partial y} = 2\,y$, $\dfrac{\partial N}{\partial x} = -1$, the differential is inexact and some relation between y and x must be assumed. (1) Let the relation be $y = 2\,x$, which is the equation of a line passing through the given end points (0, 0) and (1, 2). From this relation we have $\dfrac{dy}{dx} = 2$,

whence $$y^2\,dx - x\,dy = \left(y^2 - x\,\frac{dy}{dx}\right)dx = (4\,x^2 - 2\,x)\,dx,$$

and the integral becomes $\int_0^1 (4\,x^2 - 2\,x)\,dx = \frac{1}{3}$. (2) Let the assumed relation be $y^2 = 4\,x$, which is also satisfied by the end points (0, 0) and (1, 2); then $dx = \frac{1}{2}\,y\,dy$, and the integral becomes $\int_0^2 (\frac{1}{2}\,y^3 - \frac{1}{4}\,y^2)\,dy = 1\frac{1}{3}$. (3) If the relation is $y = 2\,x^2$, we have $dy = 4\,x\,dx$, and the integral becomes

$$\int_0^1 4\,(x^4 - x^2)\,dx = -\tfrac{8}{15}.$$

It appears therefore that the integral of an inexact differential is not determined by the initial and final values of the variables, but requires in addition a relation between those variables, that is, a path between the end points. If $dz = M\,dx + N\,dy$ is an inexact differential, dz has therefore no definite significance as a total differential so long as x and y remain independent, and assumes definiteness only when a relation between x and y is furnished. Furthermore, z is not a point function of x and y.

Returning to the differentials dQ and $d\,W$, equations (6) and (7), it is known from physical considerations that dQ is inexact, therefore Q is not a point function of T and v, and a relation between T and v must be established before the heat absorbed by the gas during a change of state can be determined. If the force acting on a moving particle is the force of gravity alone, or if it is a function of the distance of the particle from a fixed point, then it is found that the force components in (7) satisfy the relation $\dfrac{\partial X}{\partial y} = \dfrac{\partial Y}{\partial x}$. In this case $d\,W = X\,dx + Y\,dy$ is an exact differential and the work W depends only upon the end points (x_1, y_1) and (x_2, y_2). If frictional forces are taken into account, $d\,W$ is not exact

and W depends upon the path between the initial and final positions.

EXERCISES

Determine which of the following differentials are exact, and for such as are exact find the functions that produce them:

1. $e^x \sin y\, dx + e^x \cos y\, dy$.
2. $v^n\, dp + npv^{n-1}\, dv$.
3. $x^2y^5\, dx + 5\, x^3y\, dy$.
4. $\dfrac{y^2}{x}\, dx + x \log x\, dy$.
5. $(x^2 - y)\, dx - x\, dy$.
6. $(y^2 - 2\, xy - y)\, dx - (x^2 - 2\, xy + x)\, dy$.
7. $(x^2 - axy + y^2)\, dx + (y^2 + xy - ax^2)\, dy$.
8. $(\sin y - e^x y)\, dx + (x \cos y - e^x)\, dy$.

9. Show a geometrical interpretation of the differential $y\, dx - x\, dy$, and from purely geometrical considerations show that the integral $\int_{x_1, y_1}^{x_2, y_2}(y\, dx - x\, dy)$ must depend upon the path.

10. For a gas that follows the law $pv = RT$, we have $l = p$, whence $dQ = c\, dT + p\, dv$. Show that while dQ is not an exact differential, $\dfrac{dQ}{T}$ is an exact differential.

11. Find the value of the integral $\int_{0, 0}^{2, 2}(y\, dx - x\, dy)$ for the following paths between (0, 0) and (2, 2): (*a*) A path made up of the straight line joining (0, 0) to (2, 0) and the straight line joining (2, 0) to (2, 2). (*b*) A path made up of the Y-axis from (0, 0) to (0, 2) and the line joining (0, 2) to (2, 2). (*c*) The curve $y = x^3 - 4\, x$.

12. Find the value of the integral $\int_{0, 0}^{4, 3}(2\, xy\, dx + x^2\, dy)$. Show by choosing two or more paths that the integral is independent of the path.

13. If W denote the work done by an expanding gas, show that $dW = p\, dv$, that dW is an inexact differential, and that consequently W depends upon the relations of p to v during the expansion. See Art. 113.

MISCELLANEOUS EXERCISES

Find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial u}{\partial z}$ for each of the following.

1. $u = x^2y^3z^5$.
2. $u = x^3e^{2y} \cos z$.
3. $u = (x^3 + y^3 + z^3)^{\frac{1}{3}}$.
4. $u = \arctan \dfrac{x + y}{z}$.

Find $D_x y$ for the following functions.

5. $x^3y^2 - 4\, xy^4 + 3\, y^5 = 0$.
6. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$

7. $ax^2 + by^2 + 2hxy + 2ex + 2fy + g = 0.$ **8.** $\dfrac{1}{\sqrt{x^3 - y^3}} = Cy.$

Find $\dfrac{du}{dx}$ in the following cases.

9. $u = \sqrt{x^2 + y^2}, \quad y = \arctan x.$

10. $u = x^3 e^x \arcsin x.$

11. $u = \arctan \dfrac{y}{z}, \quad y = e^{-x}, \quad z = \cos x.$

12. $u = e^{ax}(y - z), \quad y = a \sin x, \quad z = \cos x.$

Form $\dfrac{\partial^3 u}{\partial x\, \partial y^2}$, $\dfrac{\partial^3 u}{\partial x^2\, \partial y}$, and $\dfrac{\partial^3 u}{\partial y^3}$ for each of the following.

13. $u = x^3 - 5x^2y + 3y^4.$ **14.** $u = e^x \cos y.$ **15.** $u = \sin x^2 + \cos xy.$

16. Show that the relation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$ holds for the following functions.

(*a*) $V = \log(x^2 + y^2)$; (*b*) $V = \arctan \dfrac{y}{x}.$

17. If $V = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, show that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0.$

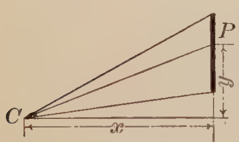

FIG. 73.

18. The potential V at a point P, in a straight line due to the attraction of a mass at an external point C, is

$$V = k \log \frac{y + \sqrt{x^2 + y^2}}{x}.$$

Find $\dfrac{\partial V}{\partial x}$ and $\dfrac{\partial V}{\partial y}$.

19. Find from van der Waals' equation

$$\left(p + \frac{a}{v^2}\right)(v - b) = BT$$

the partial derivatives $\dfrac{\partial p}{\partial T}$ and $\dfrac{\partial p}{\partial v}$. Give physical interpretations to these derivatives.

20. In an oblique triangle we have the relation $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where a, b, c denote the sides, and A the angle opposite side a. Find the rate of change of A with respect to side c, keeping sides a and b constant.

21. At the point $x = 2$, $y = -3$, on the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{25} + \dfrac{z^2}{16} = 1$, find the values of $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. Draw a figure and interpret the results geometrically.

22. If z is a homogeneous function of x and y of degree n, prove *Euler's theorem*, viz.:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

23. Verify Euler's theorem for the following functions:

(*a*) $z=(x^2+y^2)\sqrt{xy}$. (*b*) $z=x^3-3\,xy^2+2\,y^3$.

(*c*) $z=\arc\tan\frac{x}{y}$. (*d*) $z=\arc\cos\frac{x}{y}+\arc\tan\frac{y}{x}$.

24. If $z=f(y+ax)$, show that

$$(a)\ \frac{\partial z}{\partial x}=a\frac{\partial z}{\partial y};\quad (b)\ \frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}.$$

25. From the transformation equations $x=\rho\cos\theta$, $y=\rho\sin\theta$, derive the relation

$$x\,dy-y\,dx=\rho^2\,d\theta.$$

26. A point moves on the surface $z=x^2+2\,y^2$ in a vertical plane which includes the Z-axis and makes an angle of 30° with the XZ-plane. If the X-component of the point's velocity is 3 units when $x=2$, find the Y- and Z-components of the velocity.

27. Derive an approximate value for the error in computing the volume V of a cylinder from the measured height h and base radius r. Find also an expression for the relative error.

28. Find a function u of x and y that satisfies the relation

$$(a)\ \frac{\partial^2 u}{\partial x\,\partial y}=x^2+y^2;\quad (b)\ \frac{\partial^2 u}{\partial x\,\partial y}=x^3\cos y.$$

29. Given $u=f(x,y)$ and $x=\rho\cos\theta$, $y=\rho\sin\theta$; show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\rho\frac{\partial u}{\partial \rho}.$$

30. Integrate the following differentials.

$$(a)\ 2\,xy\,dx+(x^2-y^2)\,dy;\quad (b)\ 2\,x\arc\tan y\,dx+\frac{x^2-y^2}{1+y^2}\,dy.$$

31. The heat content i and entropy s respectively, of superheated steam are obtained by integrating the following equations, in which T and p are the independent variables.

$$di=(\alpha+\beta T)\,dT+Amn(n+1)p\left(1+\frac{a}{2}p\right)\frac{dT}{T^{n+1}}-\frac{Am(n+1)}{T^n}(1+ap)dp,$$

$$ds=\left(\frac{\alpha}{T}+\beta\right)dT+Amn(n+1)p\left(1+\frac{a}{2}p\right)\frac{dT}{T^{n+2}}-AB\frac{dp}{p}-\frac{Amn}{T^{n+1}}(1+ap)dp.$$

Find expressions for i and s.

CHAPTER XIV

MULTIPLE INTEGRALS. APPLICATIONS

129. Multiple integrals. In previous chapters we considered successive differentiation and successive integration of functions of a single variable and the successive partial differentiation of functions of two or more variables. We now take up the problem of successive integration of functions of several variables.

Suppose, for example, that we have given a function $f(x, y, z)$ of three independent variables. We may write

$$\int f(x, y, z)\,dz = F(x, y, z), \tag{1}$$

$$\int F(x, y, z)\,dy = F_1(x, y, z), \tag{2}$$

$$\int F_1(x, y, z)\,dx = F_2(x, y, z), \tag{3}$$

where in (1) the integration is taken with respect to z, that is, as if y and x were constants. Likewise in (2) it is taken with respect to y, and in (3) with respect to x. Substituting in (3) the value of $F_1(x, y, z)$ from (2), we have

$$F_2(x, y, z) = \int\left[\int F(x, y, z)\,dy\right]dx, \tag{4}$$

and by the use of (1) this becomes

$$F_2(x, y, z) = \int\left\{\int\left[\int f(x, y, z)\,dz\right]dy\right\}dx. \tag{5}$$

The expression (5) may be written more compactly as follows:

$$F_2(x, y, z) = \int\int\int f(x, y, z)\,dz\,dy\,dx. \tag{6}$$

It is to be understood that the integral is to be taken first with respect to z, then y, and finally x.*

An expression like the second member of (6), which indicates the result of several successive integrations, is called a **multiple integral**. If two integrations are involved, the integral is called a *double integral*; if three, a *triple integral*; etc.

The evaluation of an indefinite multiple integral differs from that of an indefinite single integral in one essential feature; namely, the form of the constant of integration. An example will illustrate this point.

Ex. Given $\int\int e^{2x}y^2\,dy\,dx$; find a function u of x and y such that

$$\frac{\partial^2 u}{\partial y\,\partial x} = e^{2x}y^2.$$

Integrating first with respect to y regarding x as a constant, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{3}e^{2x}y^3 + \text{constant of integration.}$$

Since x was considered as constant during the integration, the constant of integration may depend upon x. To make this more clear, we may observe that differentiating either

$$\tfrac{1}{3}e^{2x}y^3 + C,$$

or

$$\tfrac{1}{3}e^{2x}y^3 + \phi(x),$$

with respect to y gives the same result, $e^{2x}y^2$. Hence, we assume the more general case and write

$$\frac{\partial u}{\partial x} = \frac{1}{3}e^{2x}y^3 + \phi(x),$$

where $\phi(x)$ is an arbitrary function of x. As a special case, $\phi(x)$ may of course be a constant C. Integrating this result, keeping y constant, we obtain

$$u = \tfrac{1}{6}e^{2x}y^3 + \int\phi(x)\,dx + \psi(y).$$

Here again, since y was considered constant during the integration, the integration constant must be taken as an arbitrary function of y.

---

* Books on the calculus differ in the manner of indicating the order in which the integrals are to be taken. Some authors write the differential last which is to be taken in the first integration, etc., that is, the order of the differentials in equation (6) is exactly reversed. The above notation is adopted in this book because it shows best the manner in which the integration has arisen. In other works, the context will usually indicate to the reader the notation employed.

By assigning limits of integration to each variable, we arrive at the notion of a definite multiple integral. Thus the integral

$$\int_0^1\int_a^b e^{2x}y^2\,dy\,dx$$

indicates that $e^{2x}y^2$ is to be integrated first between the limits a and b with respect to y, and the result thus found is to be integrated with respect to x between the limits 0 and 1.

Since x is considered as constant in the integration with respect to y, the y-limits, a and b, may be functions of x. Similarly in a definite triple integral, taken first with respect to z, then y, and finally x, the z-limits may be functions of both y and x, and the y-limits functions of x.

Ex. Evaluate $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} r^2\sin\theta\,dr\,d\theta\,d\phi.$

The first integration with respect to r gives

$$\frac{1}{3}a^3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos^3\theta\sin\theta\,d\theta\,d\phi.$$

Integrating with respect to θ, we obtain

$$\frac{1}{12}a^3\int_0^{2\pi}d\phi=\frac{\pi}{6}a^3.$$

It should be observed that the upper limit of the r-integral is a function of θ

EXERCISES

Evaluate the following integrals.

1. $\displaystyle\int\int x^2y\,dy\,dx.$

2. $\displaystyle\int\int e^x\sin y\,dy\,dx.$

3. $\displaystyle\int_2^5\int_0^3\int_1^4 x^3yz^2\,dz\,dy\,dx.$

4. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\,dr\,d\theta.$

5. $\displaystyle\int_0^{\pi}\int_0^{a(1-\sin\theta)} r^2\sin\theta\,dr\,d\theta.$

6. $\displaystyle\int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a}\rho\,d\rho\,d\theta.$

7. $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\rho^3\cos^2\theta\,d\rho\,d\theta.$

8. $\displaystyle\int_0^{a}\int_0^{cz}\int_0^{\sqrt{c^2z^2-y^2}}(z^2+y^2)\,dx\,dy\,dz.$

9. $\int_0^b \int_{h_1}^{h_2} \sqrt{2\,gy}\, dy\, dx.$ 10. $\int_0^{2a} \int_0^{\text{arc cos}\frac{\rho}{2a}} \rho\, d\theta\, d\rho.$

11. $\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 y\, dy\, dx.$ 12. $\int_0^a \int_0^{2\sqrt{ax}} x^n\, dy\, dx.$

130. Plane areas by double integration, rectangular coördinates. We have seen (Art. 100) that the area bounded by a curve, the axis of x, and two ordinates is given by an integral of the form

$$\int_a^b f(x)\, dx.$$

When the area is bounded by two curves, as in Fig. 74, the area is given, not by a single integral, but by the difference of the two integrals

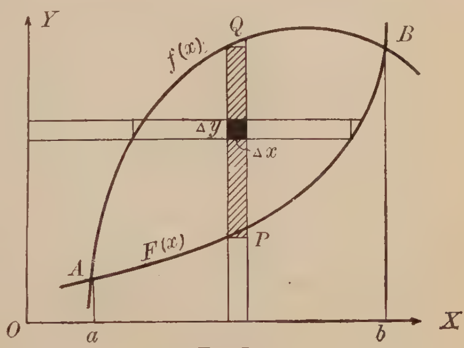

FIG. 74.

$$\int_a^b f(x)\, dx, \qquad \int_a^b F(x)\, dx.$$

The same result may be accomplished by successive integration. Consider the element of area $\Delta y\, \Delta x$. If we sum up these elements with respect to y, we shall have the area of the strip PQ, and then by summing up the strips between the limits a and b we have

$$\sum_a^b \left[\sum_P^Q \Delta y \right] \Delta x = \sum_a^b \sum_P^Q \Delta y\, \Delta x.$$

Upon passing to the limits first as $\Delta y \doteq 0$ and then as $\Delta x \doteq 0$, we have the required area given by the double integral

$$A = \int_a^b \int_{F(x)}^{f(x)} dy\, dx, \tag{1}$$

which leads to the same expression for the given area as was obtained in Art. 100.

We might equally well have summed up the elements of area

in the reverse order, namely, first with respect to x and then with respect to y. In this case, we should have obtained

$$A = \int_c^d \int_{\phi(y)}^{\psi(y)} dx\, dy, \tag{2}$$

where $\phi(y)$ and $\psi(y)$ are the inverse functions of $F(x)$, $f(x)$, respectively.

Ex. **1**. Find the area between the circle $x^2 + y^2 = a^2$ and the line $x + y = a$. See Fig 75.

The coördinates of the points of intersection are $(0, a)$ and $(a, 0)$. The lower limit of the y-integral is the value of y found from the equation $x + y = a$, namely, $y = a - x$; and the upper limit is the value of y determined from the equation $x^2 + y^2 = a^2$, namely, $y = \sqrt{a^2 - x^2}$. Hence we have

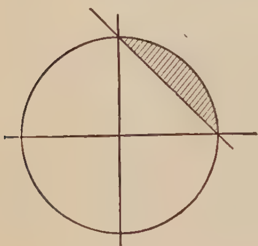

FIG. 75.

$$A = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a \left[\sqrt{a^2 - x^2} - (a - x)\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\text{arc sin}\frac{x}{a} - ax + \frac{x^2}{2}\right]_0^a$$

$$= \frac{\pi - 2}{4}a^2.$$

In some cases the abscissas of the points of intersection of the two curves may not give the proper limits of integration; in such cases it is necessary to divide the area into two or more parts.

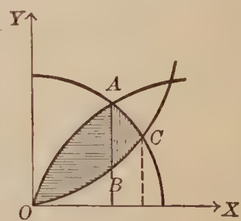

FIG. 76.

Ex. **2**. Find the area bounded by the curves $x^2 + y^2 = 25$, $y^2 = \frac{16}{3}x$, and $y = \frac{3}{16}x^2$. See Fig. 76.

The circle and the first parabola intersect at the point A whose coördinates are $(3, 4)$, and the second parabola intersects at the point C, whose coördinates are $(4, 3)$. From O to A, the equation $y^2 = \frac{16}{3}x$ gives the upper y-limit, but from A to C, this limit is determined from the equation of the circle, $x^2 + y^2 = 25$. The equation $y = \frac{3}{16}x^2$ of the lower curve OC gives the lower y-limit throughout. Hence we have

$$A = \int_0^3 \int_{\frac{3}{16}x^2}^{4\sqrt{\frac{x}{3}}} dy\, dx + \int_3^4 \int_{\frac{3}{16}x^2}^{\sqrt{25-x^2}} dy\, dx = 7.55.$$

EXERCISES

1. Find by double integration the area of a parallelogram with one side in the X-axis.

2. Find by double integration the area of a right triangle with a short side in the X-axis.

3. Find by double integration the area between the parabolas $y^2 = 8x$ and $x^2 = 8y$.

4. Find the area between the circle $x^2 + y^2 = a^2$ and the line $y = b$, $b < a$.

5. Find the area between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

6. Find the area between the curve $y = x^3$ and the Y-axis from the origin to $y = 8$.

7. Find the area bounded by the parabolas $y^2 = \frac{16}{3}x$, $y^2 = \frac{9}{4}x$ and the circle $x^2 + y^2 = 25$. Consider the first quadrant only.

8. Find the area between the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

9. Find the area bounded by the curves $x^2 + y^2 = 169$, $y^2 = \frac{144}{125}x^3$, and $5x - 12y = 0$, in first quadrant.

10. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

131. Plane areas by double integration, polar coördinates. When it is desired to find the area of a surface bounded by two curves given in polar coördinates, we may employ the method of Art. 105, or we may find the desired area by double integration. The latter method is introduced here as a simple exercise in the use of double integration.

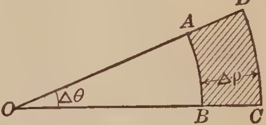

FIG. 77.

Let the polar element of area be $ABCD$, Fig. 77, bounded by the two radii OC, OD, and two circular arcs having their common center at O. Let the polar coördinates of B be (ρ, θ). From elementary geometry, we have

$$\text{sector } AOB = \tfrac{1}{2}\rho^2\Delta\theta, \tag{1}$$

$$\text{sector } DOC = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta. \tag{2}$$

Hence,
$$\Delta A = ABCD = \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta - \tfrac{1}{2}\rho^2\Delta\theta$$
$$= (\rho + \tfrac{1}{2}\Delta\rho)\Delta\theta\,\Delta\rho. \tag{3}$$

Using this polar element, we may find the area between two curves $\rho = F(\theta)$, $\rho = f(\theta)$, Fig. 78, as follows: First keep $\Delta\theta$ constant and sum the elements of area with respect to ρ. The result of this summation is an area of the type $PQSR$, the expression for which is

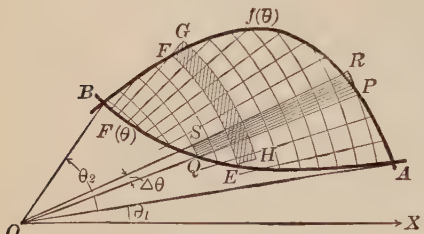

Fig. 78.

$$\Delta\theta \cdot \underset{\Delta\rho \doteq 0}{L} \sum_{OQ}^{OP} (\rho + \tfrac{1}{2}\Delta\rho)\Delta\rho$$

$$= \Delta\theta \int_{OQ}^{OP} \rho\, d\rho. \quad \text{(Art. 100)}$$

If now we sum with respect to θ, we get the sum of the wedge-like slices, and the limit of this sum is the required area. We have therefore

$$A = \underset{\Delta\theta \doteq 0}{L} \sum_{\theta_1}^{\theta_2} \Delta\theta \cdot \int_{OQ}^{OP} \rho\, d\rho = \int_{\theta_1}^{\theta_2} \int_{OQ}^{OP} \rho\, d\rho\, d\theta.$$

Replacing OQ and OP by $F(\theta)$ and $f(\theta)$ respectively, we obtain the formula

$$A = \int_{\theta_1}^{\theta_2} \int_{F(\theta)}^{f(\theta)} \rho\, d\rho\, d\theta. \quad (4)$$

The area included between the curve $\rho = f(\theta)$ and two radii, as OA and OB, is obtained from (4) by making $F(\theta) = 0$.

The required area may also be obtained as follows: Summing first with respect to θ, keeping $\Delta\rho$ constant, we obtain a segment of a circular ring of the type $EFGH$. A second summation with respect to ρ gives the sum of such ring segments, the limit of which sum is the area A. The resulting formula is

$$A = \int_{\rho_1}^{\rho_2} \int_{\phi(\rho)}^{\psi(\rho)} \rho\, d\theta\, d\rho, \quad (5)$$

where $\phi(\rho)$ and $\psi(\rho)$ are the inverse functions of $F(\theta)$ and $f(\theta)$, respectively.

Ex. Find the area between the circle $\rho = \cos\theta$ and one loop of the lemniscate $\rho^2 = \cos 2\theta$. Fig. 79.

If formula (4) is used, the area must be taken in two parts. The upper limit for both integrations is determined from the equation of the circle

$\rho = \cos\theta$. For values of θ between 0 and $\frac{\pi}{4}$, the lower limit of the ρ-integration is obtained from the equation of the lemniscate $\rho^2 = \cos 2\theta$. Since, however, no part of this loop of the lemniscate lies to the left of the line OA,

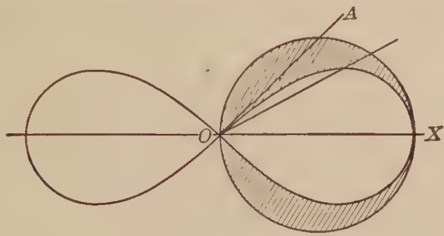

FIG. 79.

the lower limit for the ρ-integration for values of θ greater than $\frac{\pi}{4}$ is 0. Hence we have

$$A = 2\int_0^{\frac{\pi}{4}}\int_{\sqrt{\cos 2\theta}}^{\cos\theta} \rho\, d\rho\, d\theta + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\cos\theta} \rho\, d\rho\, d\theta = \frac{\pi - 2}{4}.$$

EXERCISES

1. Find by double integration the area of the circle $\rho = a$.

2. Find the area between the cardioid $\rho = 2a(1 - \cos\theta)$ and the circle $\rho = -a\cos\theta$.

3. Find the area between two circles tangent internally and having radii r_1 and r_2 respectively. Work also by single integration.

4. Find the entire area of the cardioid $\rho = 2a(1 - \cos\theta)$.

5. Find the area of one loop of the lemniscate $\rho^2 = a^2\cos 2\theta$.

6. Find the area between the circle $\rho = \cos\theta$ and one loop of the lemniscate $\rho^2 = \cos 2\theta$ by the use of formula (5).

7. Find the areas between the cardioid $\rho = 2a(1 - \cos\theta)$ and the circle $\rho = 2a$. (The shaded areas $OBAD$ and $BDCB$, Fig. 80.) Work also by single integration.

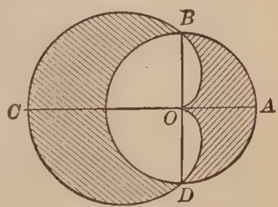

FIG. 80.

8. Work Ex. 3 by formula (5).

132. Volumes by triple integration, rectangular coördinates. Consider the volume bounded by the coördinate planes and any surface whose equation is $z = f(x, y)$, where $f(x, y)$ is a

continuous function. Through two neighboring points P and Q (Fig. 81) within the solid let planes be passed parallel to the three coördinate planes. The six planes will inclose a rectangular parallelopiped PQ whose volume is $\Delta z\,\Delta y\,\Delta x$; for if we

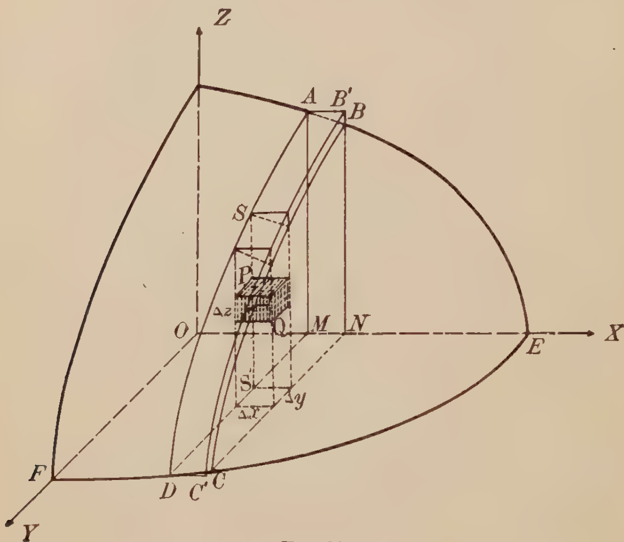

FIG. 81.

assume the coördinates of P and Q to be respectively (x, y, z) and $(x+\Delta x, y+\Delta y, z+\Delta z)$, the edges of the parallelopiped are Δx, Δy, and Δz, respectively.

We first take the sum of these elementary parallelopipeds with edges lying along the line $S'PS$. With x, y, Δx, and Δy constant, the limit of this sum as Δz is made to approach zero is the volume of the prism whose base is $\Delta x\,\Delta y$ and whose altitude is $S'S$. Now with x and Δx constant, we take the sum of the prisms lying between the planes AMD and BNC. By an extension of the method of Art. 100, it follows that as Δy is made to approach zero, this sum approaches the volume of the cylindrical slice $AB'C'DMN$. Finally, we take the sum of the slices parallel to the YZ-plane. As Δx approaches zero, the volume of the cylindrical slice approaches that of the actual slice $ABCDMN$. Hence,

the limit of the sum of these slices, as Δx approaches zero, gives the volume of the solid. We have, therefore,

$$\text{Volume of prism} = \Delta y\, \Delta x \underset{\Delta z \doteq 0}{L} \sum_0^{SS'} \Delta z = \Delta y\, \Delta x \int_0^{SS'} dz.$$

$$\text{Volume of cylindrical slice} = \Delta x \underset{\Delta y \doteq 0}{L} \sum_0^{MD} \Delta y \left(\int_0^{SS'} dz \right)$$

$$= \Delta x \int_0^{MD} \int_0^{SS'} dz\, dy.$$

$$\text{Volume of solid} = \underset{\Delta x \doteq 0}{L} \sum_0^{OE} \Delta x \left(\int_0^{MD} \int^{SS'} dy\, dz \right)$$

$$= \int_0^{OE} \int_0^{MD} \int_0^{SS'} dz\, dy\, dx. \tag{1}$$

Hence the volume is obtained by a triple integration, provided the limits of integration are properly chosen.

The first summation, namely, that with respect to z, extends from zero to SS', which is the value of the ordinate of a point on the surface. The upper limit is, therefore, a variable which is given by $z = f(x, y)$, where x, y are the coördinates of the point S'. The second summation is taken from zero to MD, and hence the limits are zero, and the value of y on the curve FDE, that is, the value of y determined from the equation $f(x, y) = 0$, it being assumed that y is a single-valued function of x. Let this value be $y = \phi(x)$. Finally, the integration which gives the sum of all the slices between O and E has for its limits zero and the constant OE ($= a$, say). We may therefore write equation (1) in the form

$$V = \int_0^{a} \int_0^{y=\phi(x)} \int_0^{z=f(x,\, y)} dz\, dy\, dx. \tag{2}$$

It will be observed that the limits for the second and third integration are the same as those that would be used in finding the area OEF, *i.e.* the projection of the given solid on the XY-plane, by means of the double integral $\int\int dy\, dx$. This fact suggests a

method of finding the required volume by double integration, namely, by the integral

$$\int_0^a \int_0^{\phi(x)} z\,dy\,dx,$$

when z is first expressed in terms of x and y.

If the given solid is not bounded by the coördinate planes, the lower limits will not be zero, as in (1) and (2); they can be readily determined, however, in the same manner as the upper limits.

EXERCISES

1. Determine the limits of integration for the triple integral $\int\int\int dz\,dy\,dx$ required in finding the volume of the pyramid bounded by the coördinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Draw the figure.

2. The cone whose equation is $y^2+z^2=c^2x^2$ has the X-axis as its axis. Determine the limits of integration when the volume of the cone is found from the triple integral $\int\int\int dz\,dy\,dx$. Also when the volume is found from the triple integral $\int\int\int dx\,dy\,dz$. Let h denote the altitude.

3. Find the volumes in Exs. 1 and 2.

4. Find by triple integration the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

5. Find the volume inclosed by the surface

$$x^2y^2+c^2z^2=a^2y^2$$

and the planes $y=0$ and $y=c$.

6. Show that the volume generated by revolving a plane figure about the X-axis may be found from the formula

$$V=2\pi\int_c^d\int_{f(x)}^{F(x)} y\,dy\,dx.$$

Compare with results in Art. 106.

7. Find the entire volume bounded by the surface whose equation is

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

8. Find the volume of the wedge cut from the cylinder $x^2+y^2=a^2$ by the planes $z=0$ and $z=x\tan\beta$.

133. Volumes by triple integration, polar coördinates. Let ρ, θ, ϕ denote the coördinates of any point P within a solid, where, as usual, ρ is the distance OP (see Fig. 82), θ is the angle ZOP which OP makes with the Z-axis, and ϕ is the angle XOP' which the projection of OP on the XY-plane makes with the X-axis. Through P suppose three surfaces passed: (1) the surface of a sphere with radius OP and O as a center; (2) a conical surface produced by revolving OP about OZ as an axis; (3) a plane surface passed through OP and OZ. Now let a second spherical surface be passed through P_1, a second conical surface through S by the rotation of OSS_1 about OZ, and a second plane surface through OZ and OQ. The six surfaces inclose a solid element $PQRSS_1R_1Q_1P_1$ having the two spherical surfaces $PQRS$ and $P_1Q_1R_1S_1$, the two conical surfaces PP_1Q_1Q and SS_1R_1R, and the plane surfaces PP_1S_1S and QQ_1R_1R. By means of such sets of surfaces the entire solid may be divided into elements of this type.

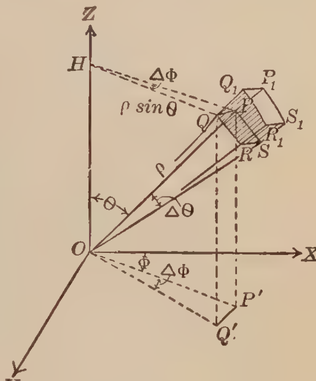

Fig. 82.

Let $PP_1 = \Delta\rho$, angle $POS = \Delta\theta$, angle $P'OQ' =$ angle $PHQ = \Delta\phi$. Then the sides of the given element of volume are $PP' = \Delta\rho$, $PQ = PH\,\Delta\phi = \rho \sin\theta\,\Delta\phi$, and $PS = \rho\,\Delta\theta$. Consequently, the volume of this element is $\rho^2 \sin\theta\,\Delta\theta\,\Delta\phi\,\Delta\rho$, plus other terms which vanish when we pass to the limit.* Therefore, denoting the required volume by V, we may write,

$$V = \underset{\Delta\theta \doteq 0}{L}\ \underset{\Delta\phi \doteq 0}{L}\ \underset{\Delta\rho \doteq 0}{L} \sum\sum\sum \rho^2 \sin\theta\,\Delta\rho\,\Delta\phi\,\Delta\theta,$$

* The exact expression for the volume element is

$$\rho^2 \sin\theta\,\Delta\rho\,\Delta\theta\,\Delta\phi \left[\frac{\sin\left(\theta + \frac{\Delta\theta}{2}\right)}{\sin\theta}\right]\left[\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\right]\left(1 + \frac{\Delta\rho}{\rho} + \frac{\Delta\rho^2}{3\,\rho^2}\right).$$

It will be seen that as $\Delta\theta \doteq 0$ the factors in the first two parentheses approach 1, and as $\Delta\rho \doteq 0$, the last factor approaches 1.

whence
$$V=\int\int\int\rho^2\sin\theta\,d\rho\,d\phi\,d\theta, \tag{1}$$

each integral being taken between the limits required by the conditions of the problem under consideration. This summation of the elements of volume can be effected in any order so long as the given volume is continuous. The method of determining the limits of integration is illustrated by the following example:

Ex. 1. Find the volume included by the surface $\rho=a\sin^2\theta\cos\phi$.

This volume lies entirely to the right of a tangent plane through the origin perpendicular to the initial line. The limits for ϕ are, therefore, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the entire volume will be included if θ be given the limits 0 and π (see Fig. 82). The limits of ρ are, of course, 0 and $a\sin^2\theta\cos\phi$. Hence we have

$$V=\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\sin^2\theta\cos\phi}\rho^2\sin\theta\,d\rho\,d\phi\,d\theta$$

$$=\frac{a^3}{3}\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7\theta\cos^3\phi\,d\phi\,d\theta=\frac{128}{315}a^3.$$

For a solid of revolution formula (1) may be simplified as follows: Taking the Z-axis as the axis of revolution, the limits of the integration with respect to ϕ are clearly 0 and 2π; hence (1) becomes

$$V=2\pi\int\int\rho^2\sin\theta\,d\rho\,d\theta. \tag{2}$$

The limits of integration for ρ and θ are precisely those that would be employed in finding the area of the plane figure revolved.

Ex. 2. Find the volume generated by revolving the cardioid

$$\rho=2a(1-\cos\theta)$$

about the initial line.

We have in this case

$$V=2\pi\int_0^\pi\int_0^{2a(1-\cos\theta)}\rho^2\sin\theta\,d\rho\,d\theta$$

$$=\frac{16\pi a^3}{3}\int_0^\pi(1-\cos\theta)^3\sin\theta\,d\theta=\frac{64}{3}\pi a^3.$$

EXERCISES

1. Find by polar coördinates the volume of a sphere (a) when the origin is taken at the center; (b) when the origin is taken at the end of a diameter.

2. By passing to polar coördinates find the entire volume of the solid included by the surface

$$(x^2 + y^2 + z^2)^2 = axyz.$$

3. In the same manner find the volume included by the surface

$$(x^2 + y^2 + z^2)^3 = 27\, a^3xyz.$$

4. Find the volume obtained by revolving the lemniscate $\rho^2 = a^2 \cos 2\,\theta$ about the initial line.

5. Find the volume obtained by revolving the loop of the curve

$$\rho = a\,\frac{\cos 2\,\theta}{\cos \theta}$$

about the initial line.

6. Find the volume obtained by revolving the curve $\rho = 2\,a \cos\theta + b$ about the initial line. First plot the curve and determine proper limits for θ.

7. Show the different types of solid elements produced when the summation is effected in the following orders:

(a) with respect to ρ, then θ, then ϕ;
(b) with respect to ρ, then ϕ, then θ;
(c) with respect to θ, then ϕ, then ρ.

134. Additional examples. In the solution of problems that involve the summation principle there are three steps: (1) The choice of an element; (2) the determination of the limits of integration so that the whole of the region involved, and only that region, shall be included; (3) the integration. The beginner is likely to encounter more difficulty in the first two processes than in the third; in other words, the setting up of the integral with proper limits of integration is usually the difficult part of the problem. In attacking such a problem, therefore, attention should first be directed to the type of element. While in most cases the rectangular and polar elements heretofore used are most convenient, it is frequently possible to choose other types that lead more directly to the desired result. It is generally preferable to take the element so that only a single integration is necessary; and frequently the polar element leads to a simpler integration than the rectangular element. Having

chosen the element, care must be taken to fix upon proper limits of integration. Sometimes it may be necessary to trace the curve representing the given function to determine these limits, but usually the limits are readily found from the conditions present.

The following problems are chosen to illustrate more fully the process of setting up the integral. They should be carefully studied.

Ex. **1.** Find the volume bounded by the surface $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ and the plane $z = c$. See Fig. 83.

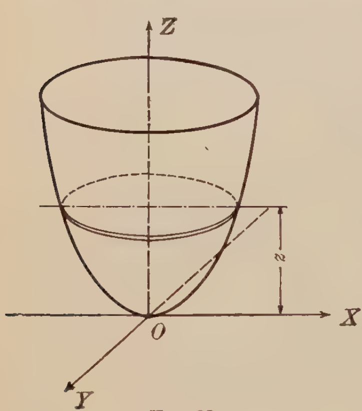

FIG. 83.

The most direct method of solution is to take the rectangular volume element $\Delta x\, \Delta y\, \Delta z$, but this choice leads to inconvenient limits of integration and a triple integral difficult to evaluate. Slight consideration of the conditions shows us that the desired volume can be obtained by the summation of slices parallel to the XY-plane. (See Art. 107.) Such slices evidently have ellipses for their boundaries, and if we denote by x', y' the semiaxes of the elliptical cross section at a distance z from the XY-plane, the volume of one of the slices is $\pi x'y'\Delta z$. From the parabolic sections $\frac{x'^2}{a} = 2z$ and $\frac{y'^2}{b} = 2z$ obtained by intersecting the surface with the XZ- and YZ-planes respectively, we have $x'y' = 2z\sqrt{ab}$, whence the volume element is $\Delta v = 2\pi\sqrt{ab}\,z\,\Delta z$. The solid extends from the plane $z = 0$ to the plane $z = c$, therefore the entire volume will be included if 0 and c are taken as the limits of integration. Hence we have the integral $\int_0^c 2\pi\sqrt{ab}\,z\,dz$, and performing the integration, we get $V = \pi c^2\sqrt{ab}$.

Ex. **2.** A plane, whose intercepts on the X-, Y-, and Z-axes are respectively a, b, and c, cuts a circular cylinder of radius r standing on the XY-plane with its axis coincident with the Z-axis. Required the volume of the quarter cylinder, Fig. 84, intercepted between the given plane and the XY-plane.

In this problem, again, the usual volume element $\Delta x\, \Delta y\, \Delta z$ may be chosen. The conditions are such, however, that another type of element is better. To form this element, let radii OE and OF be drawn in the base of the cylinder and let θ denote the angle AOF. Between these radii we take a

polar element of area $\rho\,\Delta\theta\,\Delta\rho$ and upon this as a base erect a prism meeting the oblique plane at Q. The altitude of the prism is the distance PQ between the XY-plane and the oblique plane, whose equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; hence, if x, y are the coördinates of P, then $PQ=z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$, and the volume

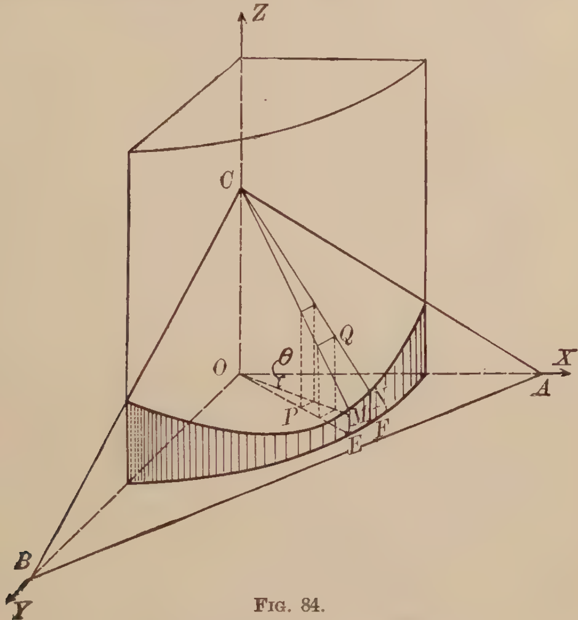

FIG. 84.

element is $z\rho\,\Delta\theta\,\Delta\rho = c\left(1-\frac{x}{a}-\frac{y}{b}\right)\rho\,\Delta\theta\,\Delta\rho$. Consider now the limits of integration. If we keep θ constant and vary ρ from $\rho=0$ to $\rho=r$, we get the volume of the wedge $OCMNFE$; then by taking the sum of such wedges between the XZ-plane ($\theta=0$) to the YZ-plane $\left(\theta=\frac{\pi}{2}\right)$, we get the required volume. Replacing x and y by $\rho\cos\theta$ and $\rho\sin\theta$, respectively, we have the volume given by the double integral

$$c\int_0^{\frac{\pi}{2}}\int_0^r\left(1-\frac{\rho}{a}\cos\theta-\frac{\rho}{b}\sin\theta\right)\rho\,d\rho\,d\theta.$$

Performing the integration, we have

$$V=r^2c\left[\frac{\pi}{4}-\frac{r}{3}\left(\frac{1}{a}+\frac{1}{b}\right)\right].$$

Ex. **3.** Required the volume of a hollow pipe bent as shown in Fig. 85, the axis of the pipe being a circular arc of radius R subtending an angle β. Take r_1 and r_2 respectively as the inner and outer radii of the cross section.

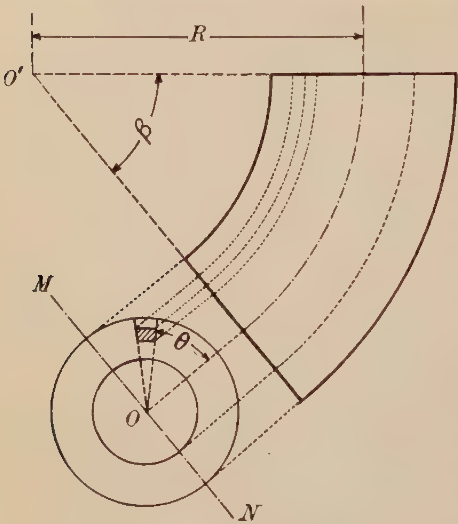

FIG. 85.

In the cross section of the pipe we choose an element of area $\rho\,\Delta\theta\,\Delta\rho$, where ρ is the distance of the element from the center O of the cross section. Evidently the distance of this element from an axis through O' is $R - \rho \sin\theta$. We may therefore take as the volume element a rod of cross section $\rho\,\Delta\theta\,\Delta\rho$ bent into an arc of radius $R - \rho\sin\theta$ and subtending an angle β. The length being $\beta(R - \rho\sin\theta)$, the volume is

$$\beta(R - \rho\sin\theta)\rho\,\Delta\theta\,\Delta\rho.$$

The limits for ρ are obviously r_1 and r_2; and one half of the required volume, that on one side of the plane MN, will be obtained by taking $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ as the limits for θ. Hence we have

$$V = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r_1}^{r_2}\beta(R - \rho\sin\theta)\,\rho\,d\theta\,d\rho = \pi\beta R(r_2^2 - r_1^2).$$

A still easier solution is given by the theorem of Pappus (Art. 137).

EXERCISES

1. In illustrative example 2, let the quarter cylinder be intersected by a sphere of radius c with its center at the origin. Using the same type of volume element, determine the limits of integration and find the volume of the part of the quarter cylinder intercepted between the sphere and the XY-plane.

2. A conoid is generated by a line kept parallel to a given plane and moved so as to keep in contact with an ellipse and with a straight line AB, Fig. 86,

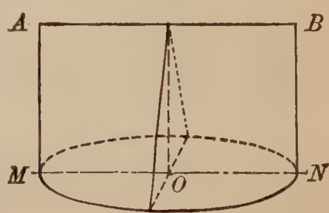

FIG. 86.

parallel to the plane of the ellipse. The semiaxes of the ellipse are a and b, and c is the distance of the line from the plane of the ellipse. Find the volume of the conoid (a) by taking elements parallel to the base; (b) by taking elements perpendicular to the line AB.

3. Find the volume $OABCD$, Fig. 87, formed by the three coördinate planes and the warped surface generated by the line EF, which remaining parallel to the XY-plane slides on the lines AD and BC.

If $OC = a$, $OD = b$, and $OA = c$, and $\Delta z\, \Delta y\, \Delta x$ is taken as the element, show that the limits of integration for z are 0 and $c\left(1 - \dfrac{ay}{b(a-x)}\right)$, and those for y are 0 and $\dfrac{b}{a}(a-x)$. Also choose an element such that a single integration is sufficient.

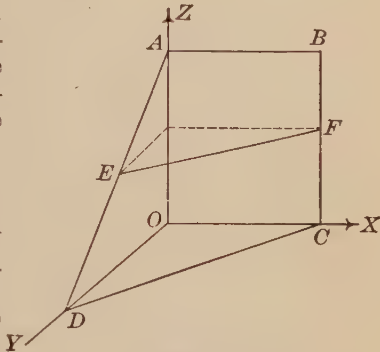

FIG. 87.

4. Show four different elements of area that may be employed in finding the area of an ellipse. Write the integrals and determine the proper limits of integration in each case.

5. Show that the volume of a solid of revolution may be found by means of the double integral $2\pi \int \int y\, dy\, dx$. What type of solid element leads to this integral? Determine the proper limits of integration.

6. Find the volume of the part cut from a sphere of radius a by a cylinder of radius $\dfrac{a}{2}$, one element of which contains the center of the sphere.

7. A right cylinder whose intersection by the XY-plane gives loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ is cut by a plane that intersects the XY-plane in the Y-axis and makes an angle of 45° with the XY-plane. Find the volume of the solid bounded by the cylindrical surface, the XY-plane, and the oblique plane.

135. Mass. Mean density. If m denote the mass and V the volume of a body, the ratio $\dfrac{m}{V}$ is called the **mean density** of the body. If we take a volume element ΔV, inclosing a point P, and denote the mass of the element by Δm, the ratio $\dfrac{\Delta m}{\Delta V}$ is the mean density of the element; and the limit of this ratio as ΔV approaches zero (still including the point P) is the **density at the**

point P. If the density is the same at all points within a body, the body is said to be **homogeneous**; otherwise, it is said to be **non-homogeneous.**

Let the mass of a non-homogeneous body be divided into elements $\Delta m_1, \Delta m_2, \cdots, \Delta m_n$, whose volumes are respectively $\Delta V_1, \Delta V_2, \cdots, \Delta V_n$. Denoting by $\gamma_1, \gamma_2, \cdots, \gamma_n$ the mean densities of these elements, we have

$$\Delta m_1 = \gamma_1 \Delta V_1,\ \Delta m_2 = \gamma_2 \Delta V_2,\ \cdots,\ \Delta m_n = \gamma_n \Delta V_n. \tag{1}$$

The mass of the body is therefore

$$m = \underset{n=\infty}{L} \sum_1^n \Delta m_k = \underset{n=\infty}{L} \sum_1^n \gamma_k \Delta V_k = \int \gamma\, dV, \tag{2}$$

and the mean density of the body, which we shall denote by $\bar{\gamma}$, is given by the equation

$$\bar{\gamma} = \frac{m}{V} = \frac{\int \gamma\, dV}{V}. \tag{3}$$

If the mass is distributed continuously over a surface, we may replace the element of volume ΔV by an element of area ΔA; and if the mass is distributed along a curve, as, for example, along a thin wire, we replace ΔV by Δs, the element of length.

The element of volume ΔV should be so chosen as to lead to the simplest integrations. Usually triple integration will be necessary, but in some cases the element may be taken in such a way that a double or even single integration is sufficient.

Ex. **1**. Find the mean density of a sphere in which the density varies as the square of the distance from the center.

Since the distance ρ of the volume element from the center determines the density, it is evident that a polar element should be chosen. From the given law we may take the density at a distance ρ from the center as $k\rho^2$, k being a constant; and for the volume element ΔV we take $\rho^2 \sin\theta\, \Delta\rho\, \Delta\phi\, \Delta\theta$. From (3) we have therefore

$$\bar{\gamma} = \frac{\int_0^\pi \int_0^{2\pi} \int_0^a k\rho^4 \sin\theta\, d\rho\, d\phi\, d\theta}{\frac{4}{3}\pi a^3} = \tfrac{3}{5} k a^2.$$

The result may also be obtained by a single integration. Since the density is constant for all points at a distance ρ from the center, we may choose

for the volume element a spherical shell of thickness $\Delta\rho$. We thus obtain $\Delta V = 4\pi\rho^2\Delta\rho$, whence

$$\bar{\gamma} = \frac{4\pi\int_0^a k\rho^4\,d\rho}{\frac{4}{3}\pi a^3} = \tfrac{3}{5}ka^2.$$

The mean density is therefore $\frac{3}{5}$ of the density at the surface.

Ex. 2. Find the mass and mean density of a thin plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, assuming that the density varies as the product xy.

In this case we naturally choose the rectangular area element $\Delta A = \Delta x\,\Delta y$. The density may be noted by kxy, k being a constant. Hence, we have

$$m = \int_0^a \int_0^{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}} kxy\,dy\,dx = \tfrac{1}{8}ka^2b^2;$$

and

$$\bar{\gamma} = \frac{\frac{1}{8}ka^2b^2}{\frac{1}{4}\pi ab} = \frac{1}{2\pi}kab.$$

EXERCISES

1. Find the mass and mean density of a semicircular plate of radius a, whose density varies as the distance from the bounding diameter. Take (a) the rectangular element of area, (b) the polar element.

2. In Ex. 1 let the density vary as the distance from the center; find the mass and mean density. Take the element of area so that only a single integration is required.

3. Find the mean density of a straight wire of length l, the density of which varies as the distance from one end.

4. Find the mass and mean density of a hemispherical solid, radius a, the density varying as the distance from the base.

5. Find the mass and mean density of a thin plate in the form of a right triangle in which the density varies as the distance from one of the short sides.

6. Given a right circular cone of height h, in which the density varies as the distance from a plane through the vertex perpendicular to the axis. Find the mean density.

136. First moments. Centroids. Let a given geometrical magnitude, line, surface, or solid, be divided in any convenient way into elements — a line into elements of length, a surface into area elements, a solid into volume elements. Let each element (Δs, ΔA, or ΔV) be multiplied by the distance of some chosen point within the element from a reference line or plane. The limit of

the sum of these products as the elements are taken smaller and smaller is called the **first moment** of the geometrical magnitude with respect to the given plane or line. We shall denote first moments by the symbol M with appropriate subscripts.

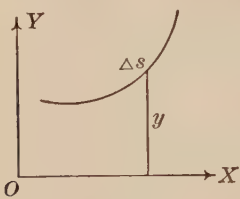

FIG. 88.

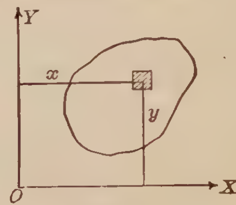

FIG. 89.

From the definition we have for the first moment M_x of a plane curve with respect to the X-axis, Fig. 88,

$$M_x = \underset{\Delta s \doteq 0}{L} \sum y \, \Delta s = \int y \, ds; \tag{1}$$

and for the first moment of a plane area with respect to the same axis, Fig. 89,

$$M_x = \underset{\Delta A \doteq 0}{L} \sum y \, \Delta A = \int y \, dA. \tag{2}$$

The first moment of a solid with respect to one of the coördinate planes, say the XY-plane (see Fig. 81), is given by the equation

$$M_{xy} = \underset{\Delta V \doteq 0}{L} \sum z \, \Delta V = \int z \, dV. \tag{3}$$

For ΔA and ΔV proper area or volume elements are to be substituted before integration is attempted. In some cases the elements may be so chosen that a single integration is sufficient, but in general double or triple integration will be required.

The first moment of the mass of a solid is derived from the corresponding moments of the geometrical magnitude by the introduction of a density factor. Thus, denoting the density by γ, the first moment with reference to the XY-plane of a mass distributed throughout a given region is

$$M_{xy} = \int \gamma z \, dV,$$

the limits of integration being taken so as to include the region in question.

Let a given solid be referred to a system of rectangular coördinates, and let M_{xy}, M_{yz}, M_{zx} denote the first moments with respect to the three coördinate planes. It is possible to find a point G whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, are given by the equations

$$\left.\begin{aligned} \bar{x} &= \frac{M_{yz}}{V} = \frac{\int x\,dV}{V} \\ \bar{y} &= \frac{M_{zx}}{V} = \frac{\int y\,dV}{V} \\ \bar{z} &= \frac{M_{xy}}{V} = \frac{\int z\,dV}{V} \end{aligned}\right\}. \tag{4}$$

The point G thus defined is called the **centroid** of the volume V.

If we replace V and dV by A and dA, formulas (4) give the centroid of a plane surface. Likewise the centroid of a curve may be obtained by using s and ds, where s denotes the length of the curve.

If in (4) we substitute the mass m of the solid for its volume V, the resulting values of $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coördinates of the centroid of the mass.* If the solid is homogeneous, the constant density factor γ, which appears in each member of the equation in the first degree, may be dropped, and in this case the centroid of the mass and that of the volume coincide. If, however, the solid is not homogeneous, γ is variable, and we have

$$m = \underset{\Delta V \doteq 0}{L} \sum \gamma \Delta V = \int \gamma\,dV,$$

$$M_{yz} = \int \gamma x\,dV, \text{ etc.,}$$

whence $$\bar{x} = \frac{\int \gamma x\,dV}{\int \gamma\,dV}, \quad \bar{y} = \frac{\int \gamma y\,dV}{\int \gamma\,dV}, \quad \bar{z} = \frac{\int \gamma z\,dV}{\int \gamma\,dV}. \tag{5}$$

The remarks of Art. 134 relative to the choice of the type of element and the determination of the limits of integration apply with equal force here. Very often the problem can be simplified by care in selecting the element.

---

* The centroid of a mass is often called the **center of gravity** of the mass.

137. General theorems relating to centroids. The following general theorems are useful in the determination of centroids.

THEOREM I. *The first moment of a volume (or mass) with respect to a plane or axis containing the centroid is zero.*

If the centroidal plane (*i.e.* plane containing the centroid) is a coördinate plane, the theorem follows at once from (4) or (5). Thus, if the centroid lies in the YZ-plane, $\bar{x}=0$, whence $M_{yz}=0$. It can be readily shown that the theorem holds for *any* centroidal plane; therefore it holds for the intersection of two such planes, that is, a centroidal axis.

THEOREM II. *A plane of symmetry of a solid or of a homogeneous mass is a centroidal plane. Likewise an axis of symmetry of a plane figure is a centroidal axis.*

Take the plane of symmetry as a coördinate plane, say the XY-plane. To an element ΔV above the plane there corresponds an equal element at the same distance below the plane. The moment of the first element with respect to the plane is $z_1\,\Delta V$, that of the second element is $-z_1\,\Delta V$, whence the moment of the pair of elements is zero. Since all the elements of the solid can be thus arranged in pairs, the first moment of the solid with respect to the plane is zero.

THEOREM III. *If V_1, V_2, $\cdots$, V_n are n volumes, and if $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\cdots$, $\bar{x}_n$ are the coördinates of their centroids, then the x-coördinate of the centroid of the system of volumes is*

$$\bar{x}=\frac{V_1\bar{x}_1+V_2\bar{x}_2+\cdots+V_n\bar{x}_n}{V_1+V_2+\cdots+V_n}. \tag{1}$$

A similar theorem holds for n masses, m_1, m_2, $\cdots$, m_n, or for n plane areas in the same plane. The numerator of the second member of (1) is the moment M_{yz} of the system with respect to the YZ-plane, and the denominator is the total volume; hence the theorem follows from (4), Art. 136.

THEOREM IV. (*Theorems of Pappus and Guldin.*)

(*a*) *If a plane curve is revolved about an axis in its plane, the area of the surface generated is equal to the product of the length of the curve and the circumference of the circle described by the centroid of the curve.*

(b) *If a plane figure is revolved about an axis in its plane, the volume of the solid of revolution generated is equal to the product of the area of the figure and the circumference of the circle (or the length of the circular arc) described by the centroid of the area.*

From (4), Art. 136, we have for a plane curve

$$\bar{y} = \frac{\int y\,ds}{s}, \text{ or } s\bar{y} = \int y\,ds,$$

where s denotes the length of the curve. The surface generated by the revolution of the curve about the X-axis is $S = 2\pi\int y\,ds$. (See Art. 110). Hence equating the two expressions for $\int y\,ds$, we obtain

$$S = 2\pi\bar{y}s. \qquad (2)$$

Evidently the theorem holds equally well for an incomplete revolution.

To prove the second theorem, let the axis of revolution be taken as the axis of x, as before; then denoting by ΔA an element of the plane area, we have for the volume generated by a complete revolution of the area,

$$V = \underset{\Delta A \doteq 0}{L} \sum 2\pi y\,\Delta A = 2\pi\int y\,dA.$$

But

$$\int y\,dA = A\bar{y};$$

hence,

$$V = 2\pi\bar{y}A. \qquad (3)$$

Ex. 1. Find the coördinates of the centroid of a circular arc of radius a which subtends an angle 2β at the center, Fig. 90.

Take the line of symmetry as the X-axis; then $\bar{y} = 0$, and the centroid lies in this line. Polar coördinates lead to the simplest integral; hence taking $x = a\cos\theta$, $\Delta s = a\,\Delta\theta$, we get,

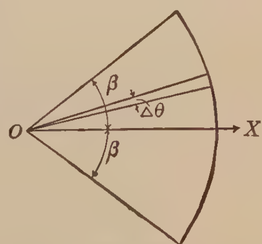

Fig. 90.

$$\bar{x} = \frac{M_y}{s} = 2\frac{\int_0^\beta a^2\cos\theta\,d\theta}{2a\beta} = \frac{2a^2\sin\beta}{2a\beta} = \frac{a\sin\beta}{\beta}.$$

For a semicircumference $\beta = \frac{\pi}{2}$, whence $\bar{x} = \frac{2a}{\pi}$.

Ex. 2. Find the coördinates of the centroid of the plate described in illustrative example 2, Art. 135.

From the defining equation, $\bar{x} = \dfrac{\int \gamma x\, dA}{\int \gamma\, dA}$, we have, upon substituting kxy for γ and $dy\, dx$ for dA,

$$\bar{x} = \frac{\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kx^2y\, dy\, dx}{\displaystyle\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} kxy\, dy\, dx} = \tfrac{8}{15}\, a.$$

Similarly, $\bar{y} = \tfrac{8}{15}\, b.$

Ex. 3. Find the centroid of a semicircle of radius a.

Taking the X-axis as the axis of symmetry, the length of the path described by the centroid as the semicircle is revolved about the Y-axis is $2\,\pi\bar{x}$. The semicircle by its revolution describes a sphere whose volume is $\frac{4}{3}\,\pi a^3$; hence by the second theorem of Pappus,

$$2\,\pi\bar{x}\cdot \tfrac{1}{2}\,\pi a^2 = \tfrac{4}{3}\,\pi a^3,$$

or

$$\bar{x} = \frac{4\,a}{3\,\pi}.$$

Ex. 4. Find the centroid of a wedge-shaped solid, Fig. 91, cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$, $\dfrac{z}{b} + \dfrac{x}{a} = 1$.

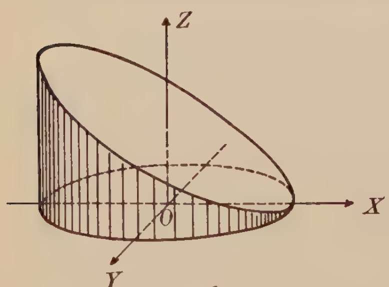

FIG. 91.

The volume of the solid is

$$\tfrac{1}{2}\,\pi a^2 \cdot 2\,b = \pi a^2 b.$$

The first moment with respect to the XY-plane is given by the integral $\int\int\int z\, dz\, dy\, dx$. The limits of integration, as determined by an inspection of the solid, are as follows: For z, 0 and $b\left(1 - \dfrac{x}{a}\right)$; for y, $-\sqrt{a^2 - x^2}$ and $\sqrt{a^2 - x^2}$; for x, $-a$ and a. Hence, we have

$$M_{xy} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{b\left(1-\frac{x}{a}\right)} z\, dz\, dy\, dx$$

$$= b^2\int_{-a}^{a}\left(1 - \frac{x}{a}\right)^2 \sqrt{a^2 - x^2}\, dx = \tfrac{5}{8}\,\pi a^2 b^2.$$

For the moment with respect to the YZ-plane, we have

$$M_{zy} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{b\left(1-\frac{x}{a}\right)} x\,dz\,dy\,dx$$

$$= 2\,b\int_{-a}^{a} x\left(1-\frac{x}{a}\right)\sqrt{a^2-x^2}\,dx = -\frac{\pi a^3 b}{4}.$$

Since the XZ-plane is a plane of symmetry, $\bar{y}=0$. The coördinates of the centroid are therefore

$$\bar{x} = -\frac{\frac{1}{4}\pi a^3 b}{\pi a^2 b} = -\tfrac{1}{4}a,\quad \bar{y}=0,\quad \bar{z} = \frac{\frac{5}{8}\pi a^2 b^2}{\pi a^2 b} = \tfrac{5}{8}b.$$

EXERCISES

Find (a) the coördinates of the centroids of the following plane curves; (b) the coordinates of the centroid of the area between each of the curves and the X-axis.

1. A semicircumference.

2. An arc of the parabola $y^2 = 4\,ax$ from $x=0$ to $x=a$.

3. An arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x=0$ to $x=a$.

4. One arch of the cycloid $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$.

5. Find the centroid of the area between the cissoid $y^2 = \frac{x^3}{a-x}$ and its asymptote $x=a$.

6. Find the centroid of a segment of the parabola $y^2 = 4\,ax$ cut off by the double ordinate corresponding to $x=a$, assuming that the density at any point is proportional to the distance from the Y-axis.

7. Find the centroid of a semicircle of radius a, assuming the density proportional to the distance from the bounding diameter.

8. Find the centroid of a segment of a homogeneous paraboloid of revolution between the origin and $x=a$.

9. Find the centroid of a hemisphere whose density varies as the distance from the base.

10. Find the coördinates of the centroid of one loop of the curve

$$\rho = a\sin 2\,\theta.$$

11. Find the coördinates of the centroid of the mass of one eighth of a sphere included between the coördinate planes, assuming that the density varies as the distance from the origin.

12. From Theorem III, Art. 137, show that if points P and Q are the centroids of two volumes V_1 and V_2 (or areas A_1 and A_2, or masses m_1 and m_2), the centroid of the whole system $V_1 + V_2$ lies on the line joining P and Q and divides it into segments inversely proportional to V_1 and V_2.

13. A solid is composed of a right circular cylinder and a right cone. See Fig. 92. Find the centroid of the solid.

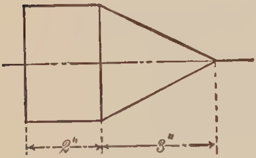

FIG. 92.

14. Three masses, $m_1 = 3$, $m_2 = 4$, and $m_3 = 5$, have their centroids located at the three vertices of an equilateral triangle the side of which has the length a. By means of Theorem III, find the position of the centroid of the system.

15. Using the Theorem of Pappus, find the centroid of the semicircumference of a circle of radius a.

16. Using the Theorem of Pappus, find the volume of the pipe described in the illustrative example 3, Art. 134.

17. Work the illustrative example 4 of this article by taking $\rho\,\Delta\theta\,\Delta\rho\,\Delta z$ as the element of volume.

138. Second moment. Radius of gyration. Let each of the elements into which a volume (or area, length, or mass) is divided be multiplied by the *square* of the distance of some chosen point in the element from a reference line or plane. The limit of the sum of these products as the elements are taken smaller is called the **second moment** * of the magnitude in question with respect to the line or plane of reference.

Formulas for second moments may therefore be derived from those for first moments by squaring the distance factor. Denoting the second moment by the general symbol I, we have the following formulas corresponding to (1), (2), and (3) of Art. 136.

For a plane curve, $$I_x = \int y^2\,ds. \tag{1}$$

For a plane area, $$I_x = \int y^2\,dA. \tag{2}$$

For a volume, $$I_{xy} = \int z^2\,dV. \tag{3}$$

* The second moment of a mass is sometimes called the **moment of inertia** of the mass.

From its definition, the second moment of a volume must have for its physical dimensions

$$\text{volume} \times (\text{length})^2;$$

hence, the second moment of a volume V can be expressed in the form

$$I = Vk^2, \tag{4}$$

in which k is a length. Similarly the second moment of an area A may be expressed in the form

$$I = Ak^2, \tag{5}$$

and likewise for any magnitude whose second moment is taken. The length k as defined by (4) or (5) is called the **radius of gyration** of the given magnitude with respect to the reference plane or axis.

Ex. **1**. Find the second moment and the radius of gyration of the area of a semicircle of radius a with respect to the bounding diameter.

Taking the origin at the center, and the bounding diameter as the axis of x, we have, by transforming to polar coördinates, $y = \rho \sin \theta$, $\Delta A = \rho\, \Delta\theta\, \Delta\rho$.

Therefore, $$I_x = \int_0^a \int_0^\pi \rho^2 \sin^2 \theta \cdot \rho\, d\theta\, d\rho = \tfrac{1}{8} \pi a^4,$$

and $$k^2 = \frac{I_x}{A} = \frac{\frac{1}{8}\pi a^4}{\frac{1}{2}\pi a^2} = \frac{1}{4} a^2,$$

whence $$k = \tfrac{1}{2} a.$$

Ex. **2**. Find the second moment of a straight rod or wire of length l about one end, assuming that the density varies as the distance from that end.

Consider the rod as lying in the X-axis with the end about which the moment is taken at the origin. At a distance x from the origin the density is kx and the mass of an element of length Δx is $kx\, \Delta x$. The second moment of this mass element about the origin is $x^2 \cdot kx\, \Delta x$; hence the second moment of the rod is

$$I = \underset{\Delta x \doteq 0}{L} \sum x^2 \cdot kx\, \Delta x = \int_0^l kx^3\, dx = \tfrac{1}{4} kl^4.$$

The mass of the rod is $m = \int_0^l kx\, dx = \tfrac{1}{2} kl^2$.

Therefore $$k^2 = \frac{I}{m} = \frac{1}{2} l^2.$$

EXERCISES

Find the second moment and the square of the radius of gyration in the following cases.

1. A rectangle of width b and length h about an axis coinciding with the short side b.

2. The same rectangle about an axis through the center parallel to the side b.

3. A circle of radius a about an axis through the center perpendicular to the plane of the circle.

4. The same circle as in Ex. 3, with the density varying as the distance from the center.

5. A triangular area of base b and height h about an axis through the vertex parallel to the base.

6. A semicircumference of a circle of radius a about the diameter.

7. The area between the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the X-axis about the X-axis.

8. The cycloidal curve about the X-axis.

9. A solid sphere of radius a with respect to a diametral plane.

10. The same sphere as in Ex. 9, assuming that the density varies as the distance from the diametral plane.

139. General theorems relating to second moments. The following general theorems are useful in the determination of second moments and radii of gyration.

THEOREM I. *If a solid (or homogeneous mass) is divided by a plane of symmetry, the second moment of the entire solid (or mass) with respect to this plane is double that of the part on one side of the plane of symmetry.*

This theorem is easily established. For any element ΔV on one side of the plane at a distance y there is a corresponding equal element on the other side at a distance $-y$. The second moment of one element is $y^2\Delta V$, that of both is $y^2\Delta V + (-y)^2\Delta V = 2y^2\Delta V$. Since all the elements of the solid can thus be arranged in pairs, the theorem follows at once. The theorem holds also for the second moment of a plane surface with respect to an axis of symmetry.

Theorem II. *The second moment of a solid (or mass) with respect to a line is the sum of the second moments with respect to two planes which intersect at right angles in the line.*

Consider the given line as the X-axis, and take the XZ- and XY-planes as the intersecting planes. A point P in a chosen volume element has the coördinates (x, y, z). Its distance from the X-axis is $\sqrt{y^2+z^2}$, and by the definition the second moment of the solid with respect to this axis is

$$I_x = \int (y^2+z^2)\,dV.$$

But $$\int y^2\,dV = I_{xz}, \text{ and } \int z^2\,dV = I_{xy};$$

hence, $$I_x = I_{xz} + I_{xy}. \tag{1}$$

A similar theorem applies to plane areas. Referring to Fig. 89, it is clear that the sum of I_x and I_y, the second moments of the plane area with respect to the axes of x and y, respectively, is the second moment of the area with respect to an axis through the origin O perpendicular to the plane of the figure.

Theorem III. *The second moment of a solid (or mass) with respect to an axis is equal to its second moment with respect to a parallel axis through the centroid plus the product of the volume (or mass) of the solid and the square of the distance between the axes.*

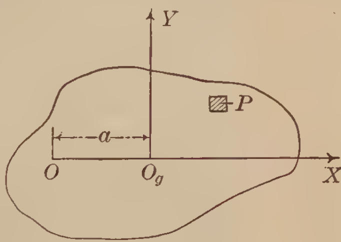

Fig. 93.

Let the given axis pierce the plane of the page at O, Fig. 93, and suppose O_g to be the trace of the parallel axis through the centroid. Take the line OO_g as the X-axis, and let a denote the distance OO_g. Referred to O_g as an origin, the point P in a given volume element has the coördinates (x, y); referred to O as origin, the coördinates are $(x+a, y)$. The second moment of the solid with respect to the axis through O_g is therefore

$$I_g = \int (x^2+y^2)\,dV,$$

while the second moment with respect to the axis through O is

$$\begin{aligned} I &= \int[(x+a)^2+y^2]\,dV \\ &= \int(x^2+y^2)\,dV + a^2\int dV + 2a\int x\,dV \\ &= I_g + a^2V + 2a\int x\,dV. \end{aligned}$$

Now $\int x\,dV$ is the first moment of the solid about the axis through O_g; hence, since this axis contains the centroid, $\int x\,dV=0$, and we have finally

$$I = I_g + a^2V. \tag{2}$$

Dividing both members of (2) by V, we get

$$k^2 = k_g{}^2 + a^2. \tag{3}$$

The student may show that the theorem holds for parallel planes, one of which contains the centroid.

Theorem III is specially useful in finding the second moments of volume, as illustrated by Ex. 2 given below.

THEOREM IV. *If $k_1, k_2, \cdots, k_n$ are the radii of gyration of n volumes (or areas, or masses) with respect to a line or plane, the radius of gyration k of the entire system with respect to the line or plane is given by*

$$k^2 = \frac{v_1k_1{}^2 + v_2k_2{}^2 + \cdots + v_nk_n{}^2}{v_1+v_2+\cdots+v_n}.$$

This theorem is useful in finding the second moment of an area or volume which can be conveniently divided into parts. The proof is similar to that of Theorem III, Art. 137.

Ex. **1**. Find the radius of gyration of a circle of radius a about a tangent.

From Theorem I, the radius of gyration about a diameter is the same as that for a semicircle; hence from Ex. 1, Art. 138, we have $k_g{}^2 = \frac{1}{4}a^2$. Using Theorem III, we obtain therefore

$$k^2 = \tfrac{1}{4}a^2 + a^2 = \tfrac{5}{4}a^2.$$

Ex. **2**. Find the second moment of a right circular cone with respect to a line through the vertex perpendicular to the axis of the solid, Fig. 94.

Take the axis of the cone as the Z-axis, and choose the vertex as the

origin. Let h be the altitude and b the radius of the base. Consider a slice of the cone parallel to the XY-plane at a distance z from it. The radius of the slice is $\frac{b}{h}z$, its thickness is Δz, and its second moment with respect to a diametral axis X', parallel to OX, is approximately

$$\frac{1}{4}\pi\frac{b^4}{h^4}z^4\,\Delta z.$$

According to Theorem III, the second moment of this slice with respect to OX is therefore

$$\frac{1}{4}\pi\frac{b^4}{h^4}z^4\,\Delta z+\pi\frac{b^2}{h^2}z^2\,\Delta z\cdot z^2$$
$$=\pi\frac{b^2}{h^2}\left(\frac{1}{4}\frac{b^2}{h^2}+1\right)z^4\,\Delta z.$$

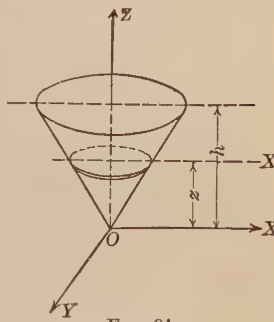

FIG. 94.

The second moment of the cone about OX is the limit of the sum of terms of this type; that is,

$$I=\pi\frac{b^2}{h^2}\left(\frac{b^2}{4h^2}+1\right)\int_0^h z^4\,dz=\left(\frac{b^2}{4h^2}+1\right)\frac{\pi b^2h^3}{5}.$$

The radius of gyration is given by the equation

$$k^2=\frac{3h^2}{5}\left(\frac{b^2}{4h^2}+1\right)=\frac{3}{20}(b^2+4h^2).$$

EXERCISES

In the following examples determine the radii of gyration.

1. A square whose side is a: (a) about an axis through the center perpendicular to its plane; (b) about a diagonal.

SUGGESTION: Make use of Theorem II.

2. An ellipse with semiaxes a and b: (a) about the major axis; (b) about the minor axis; (c) for a tangent at the end of the minor axis; (d) for a centroidal axis perpendicular to the plane of the ellipse.

3. A ring bounded by circles of radii a_1 and a_2, respectively, about a diameter. Use Theorem IV.

4. A sphere of radius a, about a diameter.

5. A right circular cone: (a) about a plane containing the axis; (b) about the axis. Take a as base radius and h as altitude.

6. A right circular cylinder: (a) about its axis; (b) about a generating element; (c) about a diameter of one base.

7. Show that the radius of gyration of a right circular cylinder about its axis is the same as that of the circular cross section about the same axis. Deduce a similar principle for prismatic solids in general.

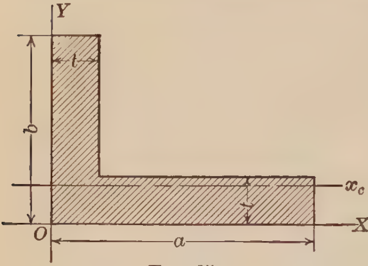

FIG. 95.

8. Find the position of the axis x_c passing through the centroid of the cross section shown in Fig. 95.

9. Determine the radius of gyration of the cross section, Fig. 95:
(*a*) with respect to the axis OX;
(*b*) with respect to the axis x_c.

140. Illustrative examples. The following illustrative problems are introduced to give the student further practice in setting up the definite integrals involved in various summations.

Ex. **1.** Find the total pressure on a circular disk of radius a held in a vertical position below the surface of a liquid.

According to the laws of hydrostatics, the intensity of liquid pressure is proportional to the distance below the liquid surface. Taking the polar element of area $\rho\,\Delta\theta\,\Delta\rho$, Fig. 96, the pressure on this element is $ky\,\rho\,\Delta\theta\,\Delta\rho$, where k is a constant; hence the total pressure is given by the double integral $k\int\int y\,\rho\,d\theta\,d\rho$ with appropriate limits of integration. Evidently the limits for ρ are 0 and a, and the area of the semicircle on one side of the vertical diameter will be included if $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are taken as the limits for θ. It should be noted that while the total pressures on the semicircles on the two sides of the vertical diameter are equal, the pressures on the upper and lower semicircles are not equal. Now taking $y = h - \rho\sin\theta$, we have for the total pressure

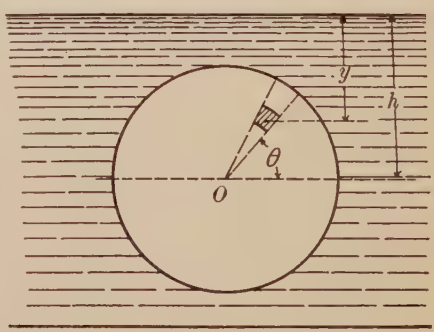

FIG. 96.

$$P = 2\,k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(h - \rho\sin\theta)\rho\,d\rho\,d\theta = \pi a^2 kh.$$

The mean intensity of pressure is therefore $\pi a^2 kh \div \pi a^2 = kh$, which is the intensity at the center of the disk.

Ex. **2.** Find the volume of liquid that will flow in one second through a rectangular orifice of width b, Fig. 97.

It is known from hydromechanics that the velocity v with which a liquid flows through a small orifice at a distance h below the liquid surface is given by the equation $v = \sqrt{2gh}$; therefore the volume flowing through an orifice having the area ΔA is $\Delta Q = \sqrt{2gh}\,\Delta A$. When the orifice is large the height h is different at different parts of the orifice and to determine the total volume flowing we must sum the volumes flowing through the elements into which the area of the orifice is divided. Thus $Q = L\sum\sqrt{2gh}\,\Delta A$. For the rectangular orifice in question we naturally take the rectangular element $\Delta x\,\Delta h$; therefore

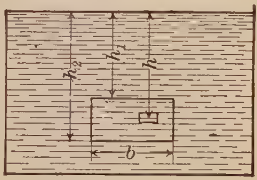

Fig. 97.

$$Q = \int_{h_1}^{h_2}\int_0^b \sqrt{2gh}\,dx\,dh = \tfrac{2}{3}b\sqrt{2g}(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}).$$

Because of contraction of the jet and friction, the discharge determined experimentally is smaller than that calculated.

EXERCISES

1. Find the total liquid pressure on a submerged vertical plate of width b. (See Fig. 97.) Find also the mean intensity of the pressure on the plate.

2. Find the liquid pressure on a vertical triangular plate having a vertex at the liquid surface and the opposite side horizontal.

3. In the preceding examples of liquid pressure, show that the total pressure is equal to the area of the plate multiplied by the intensity of pressure at the centroid of the plate. Prove that this statement holds for any submerged plane surface.

4. Write the definite integral with proper limits of integration that gives the total liquid pressure on a submerged vertical disk with an elliptical outline, the major axis being horizontal.

5. Find an expression for the volume of liquid flowing through a triangular notch, that is, a triangular orifice with its base in the liquid surface. Take b and h as the base and altitude respectively.

6. Find the volume of liquid flowing through a semicircular orifice, radius a, with the diameter in the liquid surface.

7. Set up the definite integral that gives the volume flowing through a circular orifice, radius a, the center being at a distance h below the liquid surface. Take (a) rectangular coördinates, (b) polar coördinates.

8. Find the definite integral that gives the second moment of the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with respect to an axis parallel to the Z-axis and passing through the point $(a, 0, 0)$.

SUGGESTION: Take sections parallel to the YZ-plane and use theorem III, Art. 139.

9. A cylindrical vessel is filled with water to a height h. A small orifice of area a is opened in the lower base, the water flows out, and since the water level is falling continuously, the velocity of flow varies continuously. Denoting by A the area of the cylindrical cross section, find an expression for the time required to empty the cylinder.

10. Find an expression for the time required to empty a hemispherical bowl, radius r, through an orifice of area a in the bottom.

MISCELLANEOUS EXERCISES

Find the following areas:

1. Between the parabola $y^2 = 8x$ and the circle $x^2 + y^2 = 9$.

2. The whole area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

3. The area of the loop of the curve $x^3 + y^3 = 3axy$.

4. One loop of the curve $\rho = a\sin 2\theta$.

5. The part of the curve $\rho = a\sin^3\frac{\theta}{3}$ below the X-axis.

6. The parabola $\rho = a\sec^2\frac{\theta}{2}$ between the vertex and the latus rectum.

7. Find the volume bounded by the cylinder $(x-a)^2+(y-b)^2=r^2$, the surface $xy = cz$, and the plane $z = 0$.

8. Find the volume included between the surface $a^2x^2 + b^2z^2 = 2(ax+bz)y$ and the planes $y = \pm m$.

9. Find the volume generated by revolving the curve $\rho = a(1+\cos\theta)$ about the initial line.

10. Find the area of that part of the surface of a sphere $x^2+y^2+z^2=2az$ lying within the paraboloid $z = mx^2 + ny^2$.

11. By extending the method of Art. 108, show that the length of a curve in space is given by the formula

$$s=\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2}\,dx.$$

12. From the formula of Ex. 11, find the length of the helix

$$z = a\arccos\frac{x}{b},\quad z = a\arcsin\frac{y}{b},$$

from $z = 0$ to $z = m$.

13. Find the area of the curved surface of a right cone whose base is the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, and whose altitude is c.

SUGGESTION: Taking the origin at the vertex of the cone, the equation of the surface is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \left(\frac{a}{c} z\right)^{\frac{2}{3}}$.

14. The axis of a right circular cylinder passes through the center of a sphere. The radius of the sphere is a, that of the cylinder is b, $(b < a)$. Find the volume of the part of the sphere external to the cylinder.

15. Find the volume included between the surface $xy = cz$ and the planes $x = 0$, $x = b$, $y = 0$, $y = b$, $z = 0$.

16. Find the mean density of a right circular cone whose density varies as the distance from the base.

17. Find the mean density and mass of a solid hemisphere of radius a, assuming that the density varies as the distance from a tangent plane parallel to the base.

18. Using the theorems of Pappus, find the volume and surface of a ring formed by revolving a circle of radius a about an axis at a distance b from the center of the circle, $b > a$.

19. A helical screw thread whose cross section is an equilateral triangle is cut on a cylinder. The radius of the cylinder to the root of the thread is r, and the height of the thread is a. Find by the theorem of Pappus the volume of one turn of the thread.

20. Apply the theorem of Pappus to find the surface and volume of a right circular cone.

21. Using the theorem of Pappus, show that an element of the volume of a surface of revolution is $2\pi\rho^2 \sin\phi \,\Delta\rho\, \Delta\phi$, and that the volume is therefore given by the formula

$$V = 2\pi \int\int \rho^2 \sin\phi \, d\rho \, d\phi,$$

with proper limits of integration.

22. Using the theorem of Pappus, deduce the general form for the polar volume element, viz.

$$\Delta V = \rho^2 \sin\theta \,\Delta\rho\, \Delta\phi\, \Delta\theta.$$

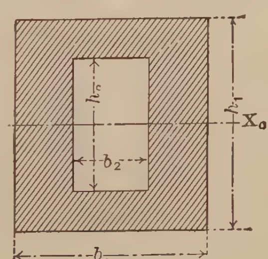

FIG. 98.

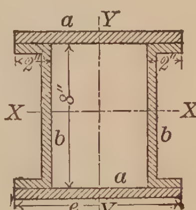

FIG. 99.

23. Find the radius of gyration of the hollow rectangle, Fig. 98, about the axis X_c, which passes through the center of the figure.

24. In Fig. 99, the thickness of the plates (a, a) and channels (b, b) is $\frac{3}{8}$ inch

throughout. Find the length e in order that the radius of gyration with respect to the axis YY shall be the same as that with respect to the axis XX.

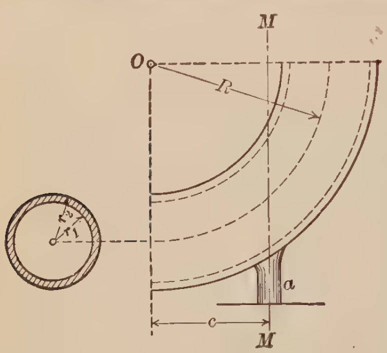

FIG. 100.

25. A pipe elbow, Fig. 100, is to be provided with a foot a to support it in the position shown. The vertical line MM through the foot should therefore pass through the centroid of the elbow. Find the distance c.

26. The following graphical method is used for finding the radius of gyration of the cross section of a hollow cylindrical column.* *Lay off to a scale of* 1 *inch equal to* 4 (*or* 40) *the inner and outer radii as the legs of a right triangle; then the hypotenuse measured to a scale of* 1 *inch equal to* 1 (*or* 10) *is the radius of gyration sought.* Give a proof of this construction.

27. By particular examples verify the following rule, due to Routh: *For homogeneous masses with axes of symmetry, the square of the radius of gyration is* $\frac{1}{3}$, $\frac{1}{4}$, *or* $\frac{1}{5}$ *of the sum of the squares of the perpendicular semi-axes, according as the mass is rectangular, elliptic, or ellipsoidal.*

28. From the second moment of a sphere about a diameter deduce by differentiation the second moment of a spherical shell about a diameter.

29. Find the centroid of the volume $OABCD$, Fig. 87.

30. (*a*) For a surface of revolution show that the radius of gyration about the axis of revolution is given by the equation

$$k^2 = \frac{\int y^3\, ds}{\int y\, ds},$$

where the X-axis is taken as the axis of revolution. (*b*) Show that for the corresponding solid of revolution

$$k^2 = \frac{\int y^4\, dx}{\int y^2\, dx}.$$

---

* B. F. LaRue in *Engineering News*, Feb. 2, 1893.

CHAPTER XV

INFINITE SERIES

141. Fundamental definitions. From the study of algebra, the student is already familiar with the elementary properties of infinite series. In the present chapter, we shall recall briefly some of the more important of those properties and develop still others.

Let $u_1, u_2, u_3, \cdots$ be an infinite succession of values following one another according to some fixed law. We denote the sum of the first n of these values by S_n, that is,

$$S_n = u_1 + u_2 + \cdots + u_n. \tag{1}$$

When n becomes infinite, we have the infinite series

$$u_1 + u_2 + u_3 + \cdots. \tag{2}$$

If S_n has a limit as n becomes infinite, that limit is called the **sum** of the infinite series. The series (2) is said to be **convergent** if we have

$$\underset{n=\infty}{L} \; S_n = A, \tag{3}$$

where A is a definite number; in all other cases the series is said to be **divergent**. As n increases indefinitely, S_n may fail to have a limit, and therefore the series may be divergent, either because S_n oscillates between two numbers, as in the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots,$$

or because S_n ultimately exceeds every finite number.

The terms of the series may be functions of some variable x. Then the series is said to converge for any particular value of this variable, say $x = x_0$, when, if x is replaced by x_0 in each term, we have

$$\underset{n=\infty}{L} \; S_n(x_0) = A.$$

If the corresponding limits exist for all values of x in a certain interval (α, β), the series is said to converge throughout the interval and to define a function in the interval. We may then write,

$$f(x) = \underset{n=\infty}{L} S_n(x), \qquad \alpha \leqq x \leqq \beta.$$

Ex. 1. Let the given series be

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots.$$

We have then

$$\underset{n=\infty}{L} S_n(x) = \underset{n=\infty}{L} \frac{1-x^n}{1-x}.$$

This limit is $\dfrac{1}{1-x}$ for all values of x within the interval $-1 < x < +1$. Hence within this interval the series defines the function

$$f(x) = \frac{1}{1-x}.$$

It is to be noted that in this case $f(x)$ is defined by the series only for the interval $(-1, 1)$, the end points being excluded. For $|x| \geqq 1$ the series becomes divergent and does not define a function.

Ex. 2. Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

This series may be written in the form

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$
$$+ \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2(n-1)}\right) + \cdots.$$

The sum of the terms in each of these parentheses is greater than $\frac{1}{2}$, and as the number m of such groups that can be formed from the given series is indefinitely large, we have

$$\underset{n=\infty}{L} S_n > \underset{m=\infty}{L} (m \cdot \tfrac{1}{2}) = \infty.$$

This result leads to an important observation; namely, that a series is not necessarily convergent because the terms themselves decrease and approach zero as n increases.

If a series containing negative terms is still convergent when all of the negative terms are taken positively, that is, when only the absolute values of the terms are considered, the series is said to converge **absolutely** or **unconditionally**. If the series is convergent when the negative terms are taken with their proper signs,

but is not convergent when those signs are taken positively, the series is said to converge **conditionally**. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots$$

is such a series. On the other hand, the series

$$1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\cdots$$

is an absolutely convergent series, since it converges when all terms are given the positive sign.

142. Tests of convergence. To test the convergence of a series, certain criteria are necessary. It is to be remembered that whether the terms of the series are constants or functions of a variable, we are concerned only with the limit of the sum of a finite number of constant terms as that number increases indefinitely. For, in case the terms are functions of a variable, we either substitute a constant value for the variable and then pass to the limit, or, what amounts to the same thing, we consider for what values of the variable the series has a limit. It is important, however, to distinguish between convergence at a point, and convergence throughout an interval.

The following tests, already considered in algebra, and consequently needing no proof here, will be found sufficient for series which arise in ordinary practice.*

THEOREM I. *Comparison test. Given a series of positive terms*

$$u_1+u_2+u_3+\cdots+u_n+\cdots.$$

If from some point on the terms of this series are never greater than the corresponding terms of a known convergent series

$$v_1+v_2+v_3+\cdots+v_n+\cdots$$

of positive terms, then the given series is convergent. If the terms of the given series are from some point on never less than the corresponding terms of a known divergent series of positive terms

$$t_1+t_2+t_3+\cdots+t_n+\cdots,$$

then the given series is divergent.

---

* Compare Rietz and Crathorne's *College Algebra*, Chapter XVI.

Ex. 1. Test the convergency of the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots. \tag{1}$$

From Ex. 1 of the previous article we have the convergent series ($|x|<1$)

$$1+x^2+x^3+x^4+\cdots+x^n+\cdots. \tag{2}$$

For $x=\frac{1}{2}$, this series becomes

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots. \tag{3}$$

Since for $n>2$, $2^{n-1}<n!$, and each term of (1) after the second is less than the corresponding term of (3); therefore the series (1) must converge.

Ex. 2. Given the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}+\cdots. \tag{4}$$

Compare the series with the series whose first n terms are

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right). \tag{5}$$

By Ex. 2 of the previous article, the series (5) is divergent. Moreover, since $2n-1<2n$, each term of the given series (4) is greater than the corresponding term of the series defined by (5); consequently, the series (4) is divergent.

THEOREM II. *Ratio test. A given series*

$$u_1+u_2+\cdots+u_{n-1}+u_n+\cdots$$

is convergent or divergent, according as the limit $\underset{n=\infty}{L}\left|\frac{u_n}{u_{n-1}}\right|$ *is less or greater than* 1.

Ex. 3. Given the series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{n-1}}{(n-1)!}+\frac{x^n}{n!}+\cdots. \tag{6}$$

We have then

$$\underset{n=\infty}{L}\left|\frac{u_n}{u_{n-1}}\right|=\underset{n=\infty}{L}\frac{\dfrac{x^n}{n!}}{\dfrac{x^{n-1}}{(n-1)!}}=\underset{n=\infty}{L}\frac{x}{n}=0,$$

for any finite value of x. Hence the series converges for all finite values of x.

It is to be noted that this second criterion applies equally well when some of the terms of the series are negative. It gives no test, however, when the limit of the ratio $\left|\frac{u_n}{u_{n-1}}\right|$ is 1; in such cases other methods must be employed.

Since each term of a series is finite, the sum of any finite number of terms is also finite, and consequently the convergency or divergency of a series is not affected by omitting a finite number of terms.

If the terms of the given series are alternately positive and negative, the following is a convenient test of convergence.

Theorem III. *An infinite series in which the terms are alternately positive and negative is convergent if its terms decrease numerically and approach zero as a limit when n increases indefinitely.**

143. Power series. By a power series we understand a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots, \tag{1}$$

where a_0, a_1, $\cdots$, a_n, etc., are constants.

Such series are of importance in mathematics because of the frequency with which they occur and because of the special properties which they possess. For example, it is not always possible to obtain the integral or the derivative of a function by integrating or differentiating term by term the series which defines the function. A power series, however, has this important property when the variable is restricted to a proper interval of convergence. Consequently, if $f(x)$ is defined by the power series

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

we may then write

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} a_0\,dx + \int_{x_0}^{x_1} a_1x\,dx + \cdots + \int_{x_0}^{x_1} a_nx^n\,dx + \cdots$$

and

$$\frac{df(x)}{dx} = \frac{d(a_0)}{dx} + \frac{d(a_1x)}{dx} + \cdots + \frac{d(a_nx)}{dx} + \cdots$$

when we place a suitable restriction upon the value of x.†

A valuable property involving the convergence of a power series is given by the following theorem.

---

* For proof of this theorem, see Rietz and Crathorne's *College Algebra*, p. 188.

† For a fuller discussion of the conditions for term-by-term differentiation and integration of a series, see *First Course*, Chap. XV.

THEOREM I. *If a power series converges for $x = \alpha$, it is absolutely convergent for all values of x such that $|x| < |\alpha|$.*

Let the given series be written in the form

$$a_0 + a_1\alpha\left(\frac{x}{\alpha}\right) + a_2\alpha^2\left(\frac{x}{\alpha}\right)^2 + \cdots + a_n\alpha^n\left(\frac{x}{\alpha}\right)^n + \cdots; \tag{2}$$

then the series of coefficients

$$a_0,\ a_1\alpha,\ a_2\alpha^2, \cdots, a_n\alpha^n, \cdots$$

must decrease indefinitely in absolute value since the given series converges for $x = \alpha$. Let M be equal to or greater than the absolute value of any number in this sequence. Then the absolute values of the terms of (2) are less than the corresponding terms of the geometric series

$$M\left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \frac{x^3}{\alpha^3} + \cdots + \frac{x^n}{\alpha^n} + \cdots\right),$$

which converges for $|x| < |\alpha|$. Hence the given series (1) converges absolutely for $|x| < |\alpha|$.

144. Maclaurin's expansion of a function in a power series. It is often convenient to express a function in terms of a series. A power series is very serviceable for this purpose, because, as already stated, such series may be differentiated and integrated term by term, thus obtaining the same result as if the operation had been performed upon the function itself. In the following article we shall discuss two methods of expanding functions in terms of a power series by making use of the principles of calculus.

Suppose we have given a function $f(x)$ which, together with its derivatives, is continuous in the vicinity of the value $x = 0$. If such a function can be represented by a power series, that series must be of the form

$$f(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n + \cdots, \tag{1}$$

where the coefficients $A_0, A_1, A_2, \cdots, A_n, \cdots$ are to be determined. Since this is a power series, we may find the successive derivatives of the function $f(x)$ by term-by-term differentiation of the series. The following identities are thus derived:

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots$$
$$f''(x) = 2A_2 + 3\cdot 2A_3x + 4\cdot 3A_4x^2 + \cdots$$
$$f'''(x) = 3\cdot 2A_3 + 4\cdot 3\cdot 2A_4x + \cdots$$
$$f^{\text{IV}}(x) = 4\cdot 3\cdot 2A_4 + \cdots$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Substituting $x = 0$ in (1) and in each of the above identities, we have

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2!\,A_2,\ f'''(0) = 3!\,A_3, \cdots.$$

Hence the successive coefficients are

$$A_0 = f(0),\ \ A_1 = f'(0),\ \ A_2 = \frac{f''(0)}{2!},\ \ A_3 = \frac{f'''(0)}{3!}, \cdots. \qquad (2)$$

We have thus the values of the unknown coefficients in the assumed expansion in terms of the successive derivatives of the given function. Substituting these values in that expansion, we have

$$\boldsymbol{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots}$$
$$\boldsymbol{+ \frac{f^n(0)}{n!}x^n + \cdots,} \qquad (3)$$

where $f^n(0)$ denotes the result obtained by differentiating the function $f(x)$ in succession n times and substituting $x = 0$ in the result. The above series is called **Maclaurin's series.**

For any given function, there still remains to be determined the interval within which the expansion obtained by (3) really represents that function. This question will be discussed in a subsequent article. Assuming that such an expansion is possible, the following examples will illustrate the method of computing the successive coefficients in the expansion. When the successive derivatives all become zero from some point on, the expansion has a finite number of terms.

Ex. 1. Expand $f(x) = (1 + x)^m$.

We have
$$f(x) = (1 + x)^m,$$
$$f'(x) = m(1 + x)^{m-1},$$
$$f''(x) = m(m-1)(1 + x)^{m-2},$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f^n(x) = m(m-1)\cdots(m-n+1)(1 + x)^{m-n},$$

whence $f(0) = 1,\ f'(0) = m,\ f''(0) = m(m-1)$, etc.

Substituting these values in (1), we have, when m is negative or fractional, the infinite series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$
$$+ \frac{m(m-1)\cdots(m-n+1)}{n!}x^n + \cdots,$$

which converges for $|x| < 1$.

If m is a positive integer, the series terminates with first $m+1$ terms, since all of the higher derivatives vanish.

Ex. 2. Expand $\sin x$ in a power series.

$$\begin{aligned} f(x) &= \sin x, & f(0) &= \sin 0 = 0,\\ f'(x) &= \cos x, & f'(0) &= 1,\\ f''(x) &= -\sin x, & f''(0) &= 0,\\ f'''(x) &= -\cos x, & f'''(0) &= -1,\\ f^{\text{IV}}(x) &= \sin x, & f^{\text{IV}}(0) &= 0,\\ &\text{etc.} & &\text{etc.} \end{aligned}$$

Hence,
$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

The interval of convergence for this series is $(-\infty, +\infty)$.

EXERCISES

Expand in power series the following functions, assuming that such expansions are possible.

1. e^x.
2. $\cos x$.
3. a^x.
4. $e^{\sin x}$.
5. $\log(1+e^x)$.
6. $\log(1+x)$.
7. $\text{arc} \sin x$.
8. $\text{arc} \tan x$.
9. $(\cos\theta)^n$.
10. $e^\theta \sec\theta$.
11. $\sec\theta$ (to four terms).
12. $e^{\text{arc} \sin \theta}$.
13. $\log(x+\sqrt{1+x^2})$.
14. $\log\cos\theta$.
15. $\frac{1}{2}(e^x + e^{-x})$.
16. $\log(1-x+x^2)$.

145. Taylor's expansion. In the last article, we studied the expansion of a given function $f(x)$ in the vicinity of the value $x=0$. The method may be easily extended to the expansion in the neighborhood of any point $x=a$, provided the given function and its successive derivatives are continuous. All that is necessary is to assume the expansion of the form

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots \quad (1)$$

and proceed precisely as in Art. 144. The resulting form of the expansion is

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$
$$+\frac{f^n(a)}{n!}(x-a)^n+\cdots, \qquad (2)$$

which holds for all values of the variable within the interval of equivalence. The series resulting from this expansion is known as **Taylor's series.**

If in (2) we replace x by $(x+a)$, we have

$$f(x+a)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^n(a)}{n!}x^n+\cdots, \qquad (3)$$

which is a form in which Taylor's series is frequently written. If x and a are interchanged, the expansion takes the form

$$f(a+x)=f(x)+f'(x)a+\frac{f''(x)}{2!}a^2+\cdots+\frac{f^n(x)}{n!}a^n+\cdots. \qquad (4)$$

Forms (3) and (4) are useful when it is desired to expand a function of the sum of two numbers in powers of one of them.

Ex. **1.** Expand e^x in powers of $x-1$.

We have

$$\begin{aligned} f(x)&=e^x, & f(1)&=e,\\ f'(x)&=e^x, & f'(1)&=e,\\ f''(x)&=e^x, & f''(1)&=e,\\ &\text{etc.} & &\text{etc.}\end{aligned}$$

Hence, $$e^x=e\left[1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots\right].$$

Ex. **2.** Express $3x^3-5x^2+8x-5$ in powers of $x-2$.

In this case

$$\begin{aligned} f(x)&=3x^3-5x^2+8x-5, & f(2)&=15,\\ f'(x)&=9x^2-10x+8, & f'(2)&=24,\\ f''(x)&=18x-10, & f''(2)&=26,\\ f'''(x)&=18, & f'''(2)&=18,\\ f^{\text{IV}}(x)&=0, & f^{\text{IV}}(2)&=0.\end{aligned}$$

Hence, we have

$$f(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3,$$

or $$3x^3-5x^2+8x-5=15+24(x-2)+13(x-2)^2+3(x-2)^3.$$

Ex. 3. Develop $\log(x+h)$ in powers of h.

We have
$$f(x+h)=\log(x+h),\qquad f(x)=\log x,$$
$$f'(x)=\frac{1}{x},\qquad f''(x)=-\frac{1}{x^2},$$
$$f'''(x)=\frac{1\cdot 2}{x^3},\qquad f^{\text{iv}}(x)=-\frac{1\cdot 2\cdot 3}{x^4},$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\qquad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f^n(x)=-(-1)^n\frac{1\cdot 2\cdots(n-1)}{x^n}.$$

Substituting in (3), we obtain
$$\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\cdots.$$

For $x=1$, we have
$$\log(1+h)=h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}+\cdots.$$

Ex. 4. Expand $\sin(x+y)$ and derive the formula
$$\sin(x+y)=\sin x\cos y+\cos x\sin y.$$

We have
$$f(x)=\sin x,$$
$$f'(x)=\cos x,$$
$$f''(x)=-\sin x,$$
$$f'''(x)=-\cos x,\ \text{etc.}$$

Substituting in (4), the result is
$$\begin{aligned}\sin(x+y)&=\sin x+y\cos x-\frac{y^2}{2\,!}\sin x-\frac{y^3}{3\,!}\cos x+\frac{y^4}{4\,!}\sin x+\frac{y^5}{5\,!}\cos x-\cdots\\&=\sin x\left(1-\frac{y^2}{2\,!}+\frac{y^4}{4\,!}-\frac{y^6}{6\,!}+\cdots\right)\\&\qquad+\cos x\left(y-\frac{y^3}{3\,!}+\frac{y^5}{5\,!}-\frac{y^7}{7\,!}+\cdots\right)\\&=\sin x\cos y+\cos x\sin y.\end{aligned}$$

Ex. 5. If $f(x)=5x^3-4x^2+18x-7$, find $f(x-2)$ by Taylor's expansion.

We have here
$$f'(x)=15x^2-8x+18,$$
$$f''(x)=30x-8,$$
$$f'''(x)=30,$$
$$f^{\text{iv}}(x)=0.$$

Hence from (1),
$$\begin{aligned}f(x-2)&=f(x)+(-2)f'(x)+\frac{(-2)^2}{2\,!}f''(x)+\frac{(-2)^3}{3\,!}f'''(x)\\&=5x^3-4x^2+18x-8\\&\qquad-30x^2+16x-36\\&\qquad\qquad+60x-16\\&\qquad\qquad\qquad-40\\&=5x^3-34x^2+94x-100.\end{aligned}$$

EXERCISES

Develop the following functions in series.

1. e^{x+h}. **2.** $(x+y)^m$. **3.** $(x+y)^6$.

4. Arc sin $(x+h)$ to four terms. **5.** Log sin $(x+h)$.

6. Find $f(x+3)$, when $f(x) = x^3 - 4x + 7$.

7. Find $f(x-1)$, when $f(x) = x^2 + 7x - 5$.

8. Show that $\log(x+\sqrt{1+x^2}) = x - \dfrac{x^3}{2 \cdot 3} + \dfrac{3x^5}{2 \cdot 4 \cdot 5} - \cdots$.

9. Expand $\sin x$ in powers of $x - a$.

10. Express $5x^3 - 6x^2 + x - 10$ in powers of $x-1$; also in powers of $x-3$. Verify the results.

11. Expand $\log x$ in powers of $x-1$. Find the interval of convergence.

12. Expand $\dfrac{1}{x}$ in powers of $x-a$ and determine the interval of convergence.

146. Taylor's theorem. Maclaurin's theorem. In discussing the expansion of a function in terms of a power series, we have assumed that the series obtained actually represents the function. We shall now see under what conditions this is true. We may write

$$f(x) = S_n(x) + R_n(x), \tag{1}$$

where $f(x)$ is the given function. If the infinite series given by $\underset{n=\infty}{L} S_n(x)$ represents $f(x)$ in a given interval, then for all values of x in that interval, we must have

$$\underset{n=\infty}{L} R_n(x) = 0; \tag{2}$$

for, $R_n(x)$ represents the difference between the given function and the sum of the first n terms of the series, that is, the error involved by stopping the expansion with n terms. It is convenient to have the value of $R_n(x)$ expressed in terms of the derivatives of $f(x)$. This value can be obtained for Taylor's expansion as follows.

From equation (1), we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x, a). \tag{3}$$

Since $R_n(x, a)$ contains the factor $\dfrac{(x-a)^n}{n!}$, we may write it in the form $\dfrac{(x-a)^n}{n!}\phi(x, a)$. Replacing $R_n(x, a)$ by this expression and transposing, we have

$$f(x)-f(a)-f'(a)(x-a)-\frac{f''(a)}{2!}(x-a)^2-\cdots-\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$
$$-\frac{\phi(x, a)}{n!}(x-a)^n=0. \tag{4}$$

To find the value of $\phi(x, a)$ in terms of the derivatives of $f(x)$, we shall consider the function

$$F(z)\equiv f(x)-f(z)-f'(z)(x-z)-\frac{f''(z)}{2!}(x-z)^2-\cdots$$
$$-\frac{f^{(n-1)}(z)}{(n-1)!}(x-z)^{n-1}-\frac{\phi(x, a)}{n!}(x-z)^n. \tag{5}$$

$F(z)$ satisfies the conditions of Rolle's theorem in that it possesses a derivative for each value of z, where $a \leqq z \leqq x$, and vanishes for $z=x$ and $z=a$. Differentiating (5) with respect to z, we have

$$F'(z)=-f'(z)+f'(z)-f''(z)(x-z)+f''(z)(x-z)-\cdots$$
$$-\frac{f^n(z)}{(n-1)!}(x-z)^{n-1}+\frac{\phi(x, a)}{(n-1)!}(x-z)^{n-1}. \tag{6}$$

The terms in the second member of this equation combine in pairs and the final result is

$$F'(z)=\frac{(x-z)^{n-1}}{(n-1)!}[\phi(x, a)-f^n(z)]. \tag{7}$$

Since $F(z)$ satisfies Rolle's theorem, $F'(z)$ vanishes for some value of z between a and x, say x_1. We have then from (7)

$$\phi(x, a)=f^n(x_1), \tag{8}$$

and

$$R_n(x)=\frac{f^n(x_1)}{n!}(x-a)^n. \tag{9}$$

Replacing $R_n(x)$ by its value as given in (9), we may now write (3) in the following form

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n-1)}(a)}{n-1}(x-a)^{n-1}$$
$$+\frac{f^n(x_1)}{n!}(x-a)^n. \qquad (10)$$

This formula is known as **Taylor's theorem.**

We are now in a position to determine the interval within which the expansion represents the given function by determining the range of values of x for which (2) holds. In the simple cases which will come under consideration this interval will usually coincide with the interval of convergence of the series.

We have seen that Maclaurin's expansion is a special case of Taylor's. By putting $a=0$, we have as the value of $R_n(x)$ in Maclaurin's series

$$R_n(x)=\frac{f^n(x_1)}{n!}x^n, \quad 0<x_1<x. \qquad (11)$$

Consequently $f(x)$ is given by the relation

$$f(x)=f(0)+f'(0)\,x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\frac{f^n(x_1)}{n!}x^n,$$

which is known as **Maclaurin's theorem.**

By means of this theorem we may determine the interval within which Maclaurin's expansion represents the function. In any given case we have only to determine the range of values of x for which $R_n\equiv\frac{f^n(x_1)}{n!}x^n$ has the limit zero as n becomes infinite.

147. Integration and differentiation of series. It is sometimes possible to expand a function into an infinite series by means of term-by-term integration or differentiation of a known series. Again if a given integral $\int f(x)\,dx$ cannot be evaluated by the ordinary exact methods of integration, it may be possible to develop $f(x)$ into an infinite series and integrate term by term. By taking a sufficient number of terms of the series resulting from the integration, we may approximate the given integral to any desired degree of accuracy. Of course the series so treated must be such as can be differentiated or integrated term by term.

Even when the function can be integrated directly it is sometimes convenient to use the method just described, for the series may be more easily handled in the subsequent operations than a complicated integral.

Ex. **1.** For $-1 < x < 1$, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Hence, $$\int_0^x \frac{dx}{1+x} = \int_0^x dx - \int_0^x x\,dx + \int_0^x x^2\,dx - \int_0^x x^3\,dx + \cdots,$$

or $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Ex. **2.** For $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Differentiating both members, we obtain

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Likewise, a second differentiation gives

$$\frac{1}{(1-x)^3} = \frac{1}{2}(1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots),$$

and in general we have

$$\frac{1}{(1-x)^m} = (1-x)^{-m} = 1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \cdots.$$

Ex. **3.** The perimeter of an ellipse, which has a for semimajor axis and e for eccentricity, is given by the integral

$$4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\phi}\,d\phi.$$

This integral can be evaluated approximately by expansion in series Thus we have

$$\sqrt{1 - e^2\sin^2\phi} = 1 - \frac{1}{2}e^2\sin^2\phi - \frac{1}{2}\cdot\frac{1}{4}e^4\sin^4\phi - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\sin^6\phi - \cdots,$$

which is a power series. Integrating term by term, we have

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2\phi}\,d\phi = \int_0^{\frac{\pi}{2}} d\phi - \frac{1}{2}e^2\int_0^{\frac{\pi}{2}} \sin^2\phi\,d\phi - \frac{1}{2}\cdot\frac{1}{4}e^4\int_0^{\frac{\pi}{2}} \sin^4\phi\,d\phi - \cdots.$$

Evaluating these integrals separately, we get for the required perimeter the expression

$$2a\pi\left[1 - \left(\frac{1}{2}\right)^2\frac{e^2}{1} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\frac{e^4}{3} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\frac{e^6}{5} - \cdots\right].$$

EXERCISES

1. From the known series

$$1 - x^2 + x^4 - x^6 + \cdots,$$

which defines the function $\frac{1}{1+x^2}$ for $-1 < x < 1$, derive the series for arc tan x.

2. For $-1 < x < 1$,

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots.$$

Derive from this relation a series for arc sin x.

3. Derive the series for $\cos x$ by differentiating that for $\sin x$.

4. Show that $\int_0^1 \frac{\log(1+x)}{x}dx = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$

5. Express $\int_0^1 \frac{\sin x\, dx}{x}$ as an infinite series.

6. Express $\int \frac{\cos x\, dx}{x}$ as an infinite series.

7. Find $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ to five figures.

8. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

9. The time of oscillation of a pendulum of length L is given by the expression

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Integrate in series, and derive an approximate expression for T when k is small.

10. Expand $\int \frac{dx}{\sqrt{\sin x}}$ in series.

11. Evaluate $\int_0^1 e^{-x^2}dx$ by expansion in series. This integral is of fundamental importance in the theory of probability.

12. Show that

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}} = \frac{\pi}{2\,ab^2}\left[1 + \frac{1}{4}k^2 - \frac{3}{64}k^4 - \frac{5}{256}k^6 - \cdots\right],$$

where $k^2 = 1 - \frac{b^2}{a^2}$.

148. Use of series in computation. Infinite series may be used advantageously in the computation of certain constants, logarithms, trigonometric functions, and roots of numbers; also in the derivation of certain useful approximations.

I. *Computation of e.*

We have $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1}$$

Therefore for $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots, \tag{2}$$

whence, taking a sufficient number of terms, $e = 2.7182818\cdots$.

II. *Computation of π.*

From the expansion

$$\text{arc}\sin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots, \tag{3}$$

which holds for $-1 < x < 1$, we get for $x = \frac{1}{2}$

$$\text{arc}\sin\frac{1}{2} = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{5} \cdot \left(\frac{1}{2}\right)^5 + \cdots, \tag{4}$$

whence $$\pi = 3.14159\cdots.$$

III. *Extraction of roots.*

We have $$(a^n \pm b)^{\frac{1}{n}} = a\left(1 \pm \frac{b}{a^n}\right)^{\frac{1}{n}} = a(1 \pm x)^{\frac{1}{n}}, \tag{5}$$

where $x = \frac{b}{a^n}$. Developing $(1 \pm x)^{\frac{1}{n}}$, we obtain

$$(1 \pm x)^{\frac{1}{n}} = 1 \pm \frac{1}{n}x - \frac{n-1}{n^2}\frac{x^2}{2!} \pm \frac{(n-1)(2n-1)}{n^3}\frac{x^3}{3!} - \cdots. \tag{6}$$

Ex. $\sqrt[5]{1000} = \sqrt[5]{1024 - 24} = 4\left(1 - \frac{3}{128}\right)^{\frac{1}{5}}$.

Substitute $\frac{3}{128}$ for x in the series

$$(1 - x)^{\frac{1}{5}} = 1 - \frac{x}{5} - \frac{4}{5}\frac{x^2}{10} - \frac{4}{5}\frac{9}{10}\frac{x^3}{15} - \cdots.$$

The result of this substitution to six figures is 0.995268; hence

$$\sqrt[5]{1000} = 4 \times 0.995268 = 3.981072.$$

IV. *Computation of logarithms.*

The series for $\log(1+x)$, that is,

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots,$$

converges slowly, and is therefore not well adapted for computation. A more useful series is derived as follows:

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

Substituting $-x$ for x, we have

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots.$$

By subtraction we get

$$\log(1+x)-\log(1-x)=\log\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right).$$

Let us take x positive and assume $x=\dfrac{1}{2y+1}$; then

$$\frac{1+x}{1-x}=\frac{y+1}{y},$$

and we have

$$\log(y+1)=\log y+2\left[\frac{1}{2y+1}+\frac{1}{3}\left(\frac{1}{2y+1}\right)^3+\frac{1}{5}\left(\frac{1}{2y+1}\right)^5+\cdots\right]. \tag{7}$$

From this series $\log(y+1)$ can be calculated when $\log y$ is known. Thus, since $\log 1=0$, we have

$$\log 2=2\left(\frac{1}{3}+\frac{1}{3}\frac{1}{3^3}+\frac{1}{5}\frac{1}{3^5}+\frac{1}{7}\frac{1}{3^7}+\cdots\right)$$

$$=0.693147\cdots.$$

$$\log 3=\log 2+2\left(\frac{1}{5}+\frac{1}{3}\frac{1}{5^3}+\frac{1}{5}\frac{1}{5^5}+\cdots\right)$$

$$=1.098612\cdots.$$

$$\log 4=2\log 2=1.386294\cdots.$$

$$\log 5=\log 4+2\left(\frac{1}{9}+\frac{1}{3}\frac{1}{9^3}+\frac{1}{5}\frac{1}{9^5}+\cdots\right)$$

$$=1.609438\cdots.$$

$$\log 6=\log 2+\log 3=1.791759\cdots.$$

Evidently it is only necessary to make the computation in the case of prime numbers. Logarithms to the base 10 are obtained from the natural logarithms by means of the following relation:

$$\log_{10} a = \log_e a \times \frac{1}{\log_e 10} = 0.4342945 \cdots \cdot \log_e a.$$

V. *Computation of trigonometric functions.*

For all values of x, we have the series

$$\left.\begin{aligned} \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}\right\}. \qquad (8)$$

These series converge rapidly, and may be used to compute the natural sine and cosine of any angle. Necessarily x must be expressed in radians.

Ex. Find the sine and cosine of $19° 30'$ correct to five figures.

We have $x = \frac{19.5}{180}\pi = .34034$. Substituting this value for x in the two series, we get

$$\sin x = 0.33381,$$
$$\cos x = 0.94264.$$

149. Approximation formulas. It is frequently convenient in computation to replace a function by another of approximately the same numerical value but having a more simple form or having a form better adapted to calculation. This substitution may be effected in many cases by expanding the given function and taking a certain number of the first terms of the series.

One of the most useful of these approximation formulas is obtained from the binomial formula. Thus let m denote a small fraction, and expand $(1 \pm m)^n$. The result is

$$(1 \pm m)^n = 1 \pm nm + \frac{n(n-1)}{2!} m^2 \pm \cdots.$$

Since m is small, powers higher than the first may be neglected, and we may write the approximate relation

$$(1 \pm m)^n = 1 \pm nm. \qquad (1)$$

An important special case is that in which $n = \frac{1}{2}$; for this case we have approximately

$$\sqrt{1 \pm m} = 1 \pm \tfrac{1}{2} m. \tag{2}$$

From (2) we have the more general formula

$$\sqrt{a^2 \pm b} = a\left(1 \pm \frac{b}{2a^2}\right), \tag{3}$$

as may be easily shown. In this formula b is small in comparison with a.

Since $e^m = 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots$, we have when m is small the approximate relation

$$e^m = 1 + m. \tag{4}$$

Similarly, taking two terms of the series for $\sin x$, $\cos x$, and $\log(1 + x)$, we obtain the approximate formulas

$$\sin m = m(1 - \tfrac{1}{6} m^2); \tag{5}$$

$$\cos m = 1 - \tfrac{1}{2} m^2; \tag{6}$$

$$\log(1 + m) = m - \tfrac{1}{2} m^2. \tag{7}$$

When m is small compared with a the following approximate relations are readily obtained.

$$\sin(a \pm m) = \sin a \pm m \cos a; \tag{8}$$

$$\log(a + m) = \log a + \frac{m}{a} - \frac{m}{2a^2}; \tag{9}$$

$$\frac{1}{a \pm m} = \frac{1}{a} \mp \frac{m}{a^2} + \frac{m^2}{a^3}. \tag{10}$$

The degree of error due to the neglected terms may be estimated by taking the maximum value of the remainder R. For example consider the approximation (5). If only two terms of the series are used, we have

$$\sin m = m - \frac{m^3}{3!} + \frac{m^5}{5!} f^{\mathrm{v}}(m_1),$$

where $0 < m_1 < m$. Since $f^{\mathrm{v}}(m_1) = \sin m_1$ cannot exceed 1, the difference between $\sin m$ and the assumed approximation $m - \frac{m^3}{6}$ cannot exceed $\frac{m^5}{120}$; hence $R \leqq \frac{m^5}{120}$. If we wish to restrict the

error to some definite limit r, we have only to write

$$|R| < r$$

and solve for m. Thus in the case just stated, if we restrict the error to one unit in the fourth decimal place, we have

$$\left|\frac{m^5}{120}\right| < .0001,$$

whence $|m| < \sqrt[5]{.012}$, or $|m| < .413$. Hence for angles lying between $-23°\ 40'$ and $+23°\ 40'$, two terms of the series give the value of $\sin m$ correct to three figures.

Ex. 1. The relation $\cos\phi = \sqrt{1 - \left(\frac{r}{l}\right)^2 \sin^2\theta}$ occurs in certain problems relating to the balancing of engines, and a simpler approximate relation is desired. Take $\frac{r}{l} = \frac{1}{6}$.

Putting $\left(\frac{r}{l}\right)^2 \sin^2\theta = m$, we have for the value of m, $\frac{1}{36}\sin^2\theta$, and this cannot exceed $\frac{1}{36}$. Hence we may use (2), and write $\cos\phi = 1 - \frac{r^2}{2\,l^2}\sin^2\theta$. The expression for R is

$$f''(m_1)\frac{m^2}{2} = -\frac{1}{8}\frac{m^2}{(1-m_1)^{\frac{3}{2}}} \text{ (since } f(m) = \sqrt{1-m}\text{)}.$$

The maximum value of m is $\frac{1}{36}$, and since $0 < m_1 < m$, the error of the approximation cannot exceed $\frac{1}{8}\frac{(\frac{1}{36})^2}{(\frac{35}{36})^{\frac{3}{2}}} = 0.0001$.

Ex. 2. In the theory of centrifugal fans the pressure ratio is given by $\frac{p_2}{p_1} = e^k$, where p_2 and p_1 denote respectively the pressure of the air entering the fan and that of the air leaving it. The exponent k is a constant depending upon the speed of the fan and is small. Taking $k = 0.04$, we have approximately $\frac{p_1}{p_2} = 1 + k = 1.04$. To determine the error, we have for the remainder $f''(k_1)\frac{k^2}{2} = e^{k_1}\frac{k^2}{2}$, the maximum value of which is

$$e^{0.04}\frac{(0.04)^2}{2} = 0.00083.$$

Ex. 3. In certain problems in surveying the relation between a circular arc and its chord is required. Let s denote the length of the arc, r the radius, and C the chord, Fig. 101. We have $s = r\alpha$, and $C = 2\,r\sin\frac{\alpha}{2}$. If α is

small the approximation (5) may be used. An approximation to C is therefore

$$C = 2r\frac{\alpha}{2}\left[1 - \frac{1}{6}\left(\frac{\alpha}{2}\right)^2\right]$$
$$= r\alpha - \tfrac{1}{24}r\alpha^3.$$

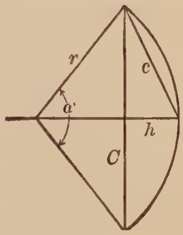

Fig. 101.

Hence,
$$s - C = \tfrac{1}{24}r\alpha^3,$$

where α is expressed in radians. If α is taken in degrees, the formula becomes

$$s - C = \frac{r\alpha^3}{4514180}.$$

The error of the approximations cannot exceed

$$2r \cdot \frac{1}{120}\left(\frac{\alpha}{2}\right)^5 = \frac{r\alpha^5}{1920}.$$

EXERCISES

1. Calculate sin 15° and cos 15° to five decimal places.

2. From the series for $\tan x$ calculate tan 12°.

3. Find $\sqrt[3]{1334}$.

4. From the logarithms given in Art. 148 calculate log 31, also log 73.

5. Using the approximate formula (10) calculate the reciprocal of 102; of 97.

6. Find the greatest value of m that will permit the approximation

$$(1 + m)^4 = 1 + 4m$$

with a maximum error of 1 in 1000.

7. Within what limits will three terms of the series for $\cos x$ give an error not exceeding 2 units in the 6th decimal place?

8. Investigate the limits of accuracy of the formula

$$\sin(a + m) = \sin a + m\cos a.$$

9. Find the length of the chord of an arc of radius 200 feet subtending an angle of 3°: (a) by trigonometric methods; (b) by the approximation formula. Compare the results and find the relative error of the approximation.

10. Derive an approximation formula for $\tan x$ and show the maximum error involved.

11. Given log 5 = 1.6094, find log 5.01 and log 5.02.

150. Maxima and minima of functions of a single variable. Taylor's expansion of a function gives a convenient method of developing the condition for maxima and minima. This method is particularly valuable where several of the derived functions

vanish for the critical value. The condition for a maximum or minimum of a function in such cases may be stated as follows:

THEOREM. *The function $f(x)$ has a maximum (or minimum) for $x=a$, if the first one of the derived functions $f'(x)$, $f''(x)\cdots$ that does not vanish for $x=a$, is of even order and negative (or positive).*

We have from Taylor's theorem,

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2!}h^2+\frac{f'''(x)}{3!}h^3+\cdots+\frac{f^n(x+\theta h)}{n!}h^n,$$

and

$$f(x-h)=f(x)-f'(x)h+\frac{f''(x)}{2!}h^2-\frac{f'''(x)}{3!}h^3+\cdots+\frac{f^n(x+\theta h)}{n!}h^n.$$

Replacing x by a, we have, after transposing the first term of the second member of the identity,

$$f(a+h)-f(a)=f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\cdots$$
$$+\frac{f^n(a+\theta h)}{n!}h^n, \qquad (1)$$

$$f(a-h)-f(a)=-f'(a)h+\frac{f''(a)}{2!}h^2-\frac{f'''(a)}{3!}h^3+\cdots$$
$$+(-1)^n\frac{f^n(a+\theta h)}{n!}h^n. \qquad (2)$$

If for $x=a$ the given function has a maximum, then $f(a)$ must exceed the value of the function for all values of the variable in the neighborhood of a; in other words, the left-hand member of both (1) and (2) must be negative for all values of h sufficiently small. The value of h can be taken so small that $hf'(a)$ will be numerically greater than the sum of the remaining terms of the second member. However $f(a+h)-f(a)$ and $f(a-h)-f(a)$ cannot both be negative unless $f'(a)=0$ and $f''(a)$ is negative, assuming that $f''(a)$ does not vanish. It may, however, happen that both $f'(a)$ and $f''(a)$ vanish. In this case, in order to have $f(a+h)-f(a)$ and $f(a-h)-f(a)$ negative, $f'''(a)$ must vanish and $f^{\text{IV}}(a)$ must be negative. In general, in order that $f(x)$ shall have a maximum value for $x=a$, the first derivative that does not vanish must be of even order and negative.

In order that $f(x)$ shall have a minimum for $x=a$, the two expressions $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be positive.

This requires that the first derivative that does not vanish shall be even and positive. The argument is similar to that given above and is left to the student.

Ex. Examine the function $2\cos x + e^x + e^{-x}$ for maxima and minima.

We have
$$f(x) = 2\cos x + e^x + e^{-x};$$
$$f'(x) = -2\sin x + e^x - e^{-x}.$$

For $x = 0$, $f'(x) = 0$; hence $x = 0$ is a critical value. We have further
$$f''(x) = -2\cos x + e^x + e^{-x}, \text{ whence } f''(0) = 0,$$
$$f'''(x) = 2\sin x + e^x - e^{-x}, \text{ whence } f'''(0) = 0,$$
$$f^{\text{iv}}(x) = 2\cos x + e^x + e^{-x}, \text{ whence } f^{\text{iv}}(0) = 4.$$

Since the fourth derivative is positive, and is the first that does not vanish, it follows that $f(x)$ is a minimum for $x = 0$.

EXERCISES

Examine for maxima and minima the following functions.

1. $\tan^2 x - 2\tan x$.

2. $e^x - e^{-x} - 2\sin x$.

3. $\sin x(1 + \cos x)$.

4. $pe^{ax} + qe^{-ax}$.

5. $x\sin x$.

6. $3\cos\theta + \tan^2\theta$.

7. $\cos x - \log\cos x$.

8. $x^{-m}e^{mx}$.

9. $e^x + e^{-x} - x^2$.

10. $\cos x(2 - \cos x)$.

151. Evaluation of indeterminate forms. Algebraic methods of evaluating certain indeterminate forms were shown in the examples of Art. 15. For the form $\frac{0}{0}$, to which all other forms may be reduced, the differential calculus furnishes a general method of evaluation, which is developed as follows:

Let the given function be of the form $\frac{f(x)}{\phi(x)}$, which reduces to the form $\frac{0}{0}$ for $x = a$. The value of the limit $\underset{x \doteq a}{L} \frac{f(x)}{\phi(x)}$ is required. Expanding each term of this fraction by Taylor's theorem, we have

$$\frac{f(x)}{\phi(x)} = \frac{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(x_1)}{n!}(x-a)^n}{\phi(a) + \phi'(a)(x-a) + \frac{\phi''(a)}{2!}(x-a)^2 + \cdots + \frac{\phi^n(x_2)}{n!}(x-a)^n}. \quad (1)$$

By hypothesis, $f(a)$ and $\phi(a)$ are each equal to zero. The above relation therefore reduces to

$$\frac{f(x)}{\phi(x)}=\frac{f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^n(x_1)}{n!}(x-a)^n}{\phi'(a)(x-a)+\frac{\phi''(a)}{2!}(x-a)^2+\cdots+\frac{\phi^n(x_2)}{n!}(x-a)^n}. \tag{2}$$

Dividing both terms of this fraction by $(x-a)$ and passing to the limit, we have

$$\underset{x \doteq a}{L}\frac{f(x)}{\phi(x)}=\frac{f'(a)}{\phi'(a)}. \tag{3}$$

If $f'(a)=0$ and $\phi'(a)\neq 0$, this limit reduces to zero; if $f(a)\neq 0$ and $\phi'(a)=0$, it becomes infinite.

If $f'(a)=0$ and $\phi'(a)=0$, the limiting value of the given function can be obtained by dividing the terms of the expanded form of the fraction $(x-a)^2$ and then passing to the limit. The result is

$$\underset{x \doteq a}{L}\frac{f(x)}{\phi(x)}=\frac{f''(a)}{\phi''(a)}. \tag{4}$$

Similarly, if $f''(a)$ and $\phi''(a)$ are both zero, we divide by $(x-a)^3$ and again take the limit, and so on.

We may therefore state the general law of procedure as follows:

To evaluate the indeterminate form $\frac{0}{0}$, *differentiate the numerator and the denominator of the given fraction and substitute the critical value of the variable in the result.*

The function $\frac{f(x)}{\phi(x)}$ may also assume the indeterminate form $\frac{0}{0}$ when x becomes infinite. The limiting value may still be found by considering $\frac{f'(x)}{\phi'(x)}$; for, we have upon putting $x=\frac{1}{z}$,

$$\underset{x \doteq \infty}{L}\frac{f(x)}{\phi(x)}=\underset{z \doteq 0}{L}\frac{-f'\left(\frac{1}{z}\right)\frac{1}{z^2}}{-\phi'\left(\frac{1}{z}\right)\frac{1}{z^2}}=\underset{z \doteq 0}{L}\cdot\frac{f'\left(\frac{1}{z}\right)}{\phi'\left(\frac{1}{z}\right)}=\underset{x \doteq \infty}{L}\frac{f'(x)}{\phi'(x)}. \tag{5}$$

Form $\frac{\infty}{\infty}$. When the function $\frac{f(x)}{\phi(x)}$ takes the form $\frac{\infty}{\infty}$, it can

be reduced to the form $\frac{0}{0}$, by writing it in the form $\frac{\frac{1}{\phi(x)}}{\frac{1}{f(x)}}$. This form can then be evaluated according to the law just stated.

It may be shown as in the case of the form $\frac{0}{0}$ that if $\frac{f'(x)}{\phi'(x)}$ has a limit as x approaches a definite number or becomes infinite, then $\frac{f(x)}{\phi(x)}$ converges to the same limit.* This principle often affords a convenient method of evaluating this indeterminate form; for we need only to differentiate the numerator and the denominator and then pass to the required limit.

Form $\infty - \infty$. When a function takes the indeterminate form $\infty - \infty$ it may be reduced to the fundamental form $\frac{0}{0}$ by writing it as follows:

$$f(x) - \phi(x) = \frac{1}{\frac{1}{f(x)}} - \frac{1}{\frac{1}{\phi(x)}} = \frac{\frac{1}{\phi(x)} - \frac{1}{f(x)}}{\frac{1}{f(x) \cdot \phi(x)}}. \tag{6}$$

Often, however, a simpler transformation will reduce the function to one of the forms $\frac{0}{0}, \frac{\infty}{\infty}$. No general rule can be given, but that transformation should be selected which gives the simplest form.

Form $0 \times \infty$. When a function $f(x) \cdot \phi(x)$ takes the form $0 \times \infty$ for $x = a$, it may be reduced to the type $\frac{0}{0}$ by writing it in the form

$$f(x) \cdot \phi(x) = \frac{f(x)}{\frac{1}{\phi(x)}}. \tag{7}$$

Forms $0^0, \infty^0, 1^\infty$. The indeterminate forms $0^0, \infty^0, 1^\infty$ arise from a function of the form $[f(x)]^{\phi(x)}$. This function may be reduced to the type form $\frac{0}{0}$ as follows:

---

* See Pierpont's *Theory of Functions*, Vol. I., p. 305. The special student of mathematics would do well to read Arts. 455–459 in the same volume.

Let $$y=[f(x)]^{\phi(x)},$$
whence $$\log y=\phi(x)\cdot\log[f(x)]. \tag{8}$$
Since, for each of the given forms, (8) takes the form $0\times\infty$, the solution is effected by (7).

Ex. 1. Evaluate $\dfrac{\tan x-\sin x}{x^3}$, for $x=0$.

We have
$$\frac{f'(x)}{\phi'(x)}=\frac{\sec^2 x-\cos x}{3x^2}\Bigg]_{x=0}=\frac{0}{0};$$
$$\frac{f''(x)}{\phi''(x)}=\frac{2\sec^2 x\tan x+\sin x}{6x}\Bigg]_{x=0}=\frac{0}{0};$$
$$\frac{f'''(x)}{\phi'''(x)}=\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}\Bigg]_{x=0}=\frac{1}{2}.$$
Hence, $$\underset{x\doteq 0}{L}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}.$$

Ex. 2. Evaluate $\sec\frac{1}{2}\pi x\cdot\log\frac{1}{x}$ for $x=1$.

This function takes the form $0\times\infty$; hence, we write it in the form
$$\frac{\log\dfrac{1}{x}}{\dfrac{1}{\sec\frac{1}{2}\pi x}}=\frac{\log\dfrac{1}{x}}{\cos\frac{1}{2}\pi x},$$
which for $x=1$ takes the form $\frac{0}{0}$. Differentiating the numerator and the denominator, we get
$$\frac{\dfrac{d}{dx}\log\dfrac{1}{x}}{\dfrac{d}{dx}\cos\dfrac{1}{2}\pi x}\Bigg]_{x=1}=\frac{-\dfrac{1}{x}}{-\dfrac{1}{2}\pi\sin\dfrac{1}{2}\pi x}\Bigg]_{x=1}=\frac{2}{\pi}.$$

Ex. 3. Evaluate $x^{\frac{1}{1-x}}$ for $x=1$.

Let $y=x^{\frac{1}{1-x}}$; then $\log y=\dfrac{1}{1-x}\log x=\dfrac{\log x}{1-x}$, which has the form $\frac{0}{0}$

Hence, by the general rule
$$\underset{x\doteq 1}{L}\log y=\frac{\dfrac{d}{dx}\log x}{\dfrac{d}{dx}(1-x)}\Bigg]_{x=1}=\frac{\dfrac{1}{x}}{-1}\Bigg]_{x=1}=-1,$$
whence $$\underset{x\doteq 1}{L}y=\frac{1}{e}.$$

EXERCISES

Evaluate the following indeterminate forms:

1. $\left.\dfrac{x^2+4x-21}{x^3-3x^2-4x+12}\right]_{x=3}$.

2. $\left.\dfrac{x^3+2x^2-x-2}{x^2+10x+16}\right]_{x=-2}$.

3. $\left.\dfrac{1-\cos x}{x^2}\right]_{x=0}$.

4. $\left.\dfrac{1}{x}(a^x-b^x)\right]_{x=0}$.

5. $\sec x-\tan x]_{x=\frac{\pi}{2}}$.

6. $\left.\dfrac{\log x}{\sqrt{1-x}}\right]_{x=1}$.

7. $\left.\dfrac{\tan\theta-\theta}{\theta-\sin\theta}\right]_{\theta=0}$.

8. $\left.\dfrac{x^x-x}{1-x+\log x}\right]_{x=1}$.

9. $\left.\dfrac{1}{\sin^2 x}-\dfrac{1}{x^2}\right]_{x=0}$.

10. $x(a^{\frac{1}{x}}-1)]_{x=\infty}$.

11. $\left.e^x\sin\dfrac{1}{x}\right]_{x=\infty}$.

12. $\left.\dfrac{2}{x}-\cot\dfrac{x}{2}\right]_{x=0}$.

13. $\sin x\log\cot x]_{x=0}$.

14. $(\sin\theta)^{\tan\theta}]_{\theta=\frac{\pi}{2}}$.

15. $x^x]_{x=0}$.

16. $(1+x^2)^{\frac{1}{x}}]_{x=0}$.

17. $(e^x+x)^{\frac{1}{x}}]_{x=0}$.

18. $\sec 2\theta\cos 5\theta]_{\theta=\frac{\pi}{2}}$.

19. $\left.\dfrac{\tan x-x}{x^3}\right]_{x=0}$.

20. $\left.\dfrac{\text{arc}\tan x-x}{x^3}\right]_{x=0}$.

21. $\left.\dfrac{e^x\sec x-1}{x}\right]_{x=0}$.

22. $\left.\dfrac{\sec\dfrac{\pi x}{2}}{\log(1-x)}\right]_{x=1}$.

23. $\left.\dfrac{x}{\sin^3 x}-\dfrac{1}{\tan^2 x}\right]_{x=0}$.

24. $\left.\dfrac{x}{x-1}-\dfrac{1}{\log x}\right]_{x=1}$.

152. Analytic condition for a singular point. In Art. 89 attention was called to certain points of plane curves, called singular points, where the derivative $\dfrac{dy}{dx}$ has not a single determinate value. At such a point $\dfrac{dy}{dx}$ must have therefore the indeterminate form $\dfrac{0}{0}$. Having now a general method of evaluating this indeterminate form, we can deduce an analytic method of dealing with singular points.

Let the equation of the curve, written in the implicit form and without radicals, be

$$f(x, y) = 0. \tag{1}$$

Then for a singular point, we have

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = \frac{0}{0}; \tag{2}$$

that is, in addition to $f(x, y) = 0$, we must have $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$. Solving these equations simultaneously, we find the coördinates of points at which singularities occur. Having found such points we may determine the character of the singularity at any one of the points by evaluating the indeterminate expression for $\dfrac{\partial y}{\partial x}$ by the methods already developed. The following examples will serve to illustrate the method of procedure.

Ex. **1**. Examine for singular points the curve

$$f(x, y) = 4x^3 - 12x^2 + 10x + xy + 11y - 3y^2 - 14 = 0.$$

We have
$$\frac{\partial y}{\partial x} = 12x^2 - 24x + 10 + y, \quad \frac{\partial f}{\partial y} = x + 11 - 6y.$$

The values $x = 1$, $y = 2$ satisfy the equations

$$f(x, y) = 0, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0;$$

hence the point (1, 2) is a singular point. The character of the singularity can be determined by evaluating the indeterminate form $\dfrac{dy}{dx}$. For convenience putting $\dfrac{dy}{dx} = p$, we have

$$p = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{12x^2 - 24x + 10 + y}{x + 11 - 6y}.$$

The right-hand member of this equation takes the form $\dfrac{0}{0}$ for $x = 1$, $y = 2$; hence to determine its value at this point, we differentiate both the numerator and the denominator with respect to x and have

$$p = -\frac{24x - 24 + p}{1 - 6p}\Bigg]_{1,2},$$

or
$$p = \frac{-p}{1-6p},$$
whence
$$2p - 6p^2 = 0,$$
$$p = 0, \text{ or } \tfrac{1}{3}.$$

Therefore at the point in question, there are two distinct tangents to the curve and consequently two branches of the curve pass through that point and the singularity is a double point. The slopes of the two tangents are 0 and $\frac{1}{3}$, respectively.

EX. 2. Examine for singular points the curve $x^4 + x^3y - 4x^2y + y^3 = 0$.

We have for the partial derivatives

$$\frac{\partial f}{\partial x} = 4x^3 + 3x^2y - 8xy, \quad \frac{\partial f}{\partial y} = x^3 - 4x^2 + 3y^2.$$

The values $x = 0$, $x = 0$ satisfy the equations $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$; hence the origin is a singular point. Putting $\frac{\partial y}{\partial x} = p$, we have for the point (0, 0)

$$p = -\left.\frac{4x^3 + 3x^2y - 8xy}{x^3 - 4x^2 + 3y^2}\right]_{0,0} = \frac{0}{0},$$

whence differentiating numerator and denominator with respect to x, we obtain

$$p = -\left.\frac{12x^2 + 6xy + 3x^2p - 8y - 8xp}{3x^2 - 8x + 6yp}\right]_{0,0}.$$

For $x = 0$, $y = 0$ the second member again takes the form $\frac{0}{0}$; hence differentiating numerator and denominator a second time, we have

$$p = -\left.\frac{24x + 6y + 6xp + 6xp - 8p - 8p}{6x - 8 + 6p^2}\right]_{0,0} = \frac{8p}{3p^2 - 4}.$$

Therefore $p(3p^2 - 4) = 8p$, and $p = 0, 2, -2$. The origin is a triple point, and the three tangents have respectively the slopes 0, 2, and -2.

EXERCISES

By the general method of this article examine the following curves for singular points. Additional exercises are furnished by the examples of Art. 89.

1. $x^4 - 4x^2y + y^3 = 0$.
2. $x^3 - 3x^2 - 3y^2 + y^3 = 0$.
3. $x^5 - 4x^4 + y^3 - 4x^2y = 0$.
4. $x^3 + 2x^2 - 4xy + 2y^2 = 0$.
5. $x^4 - x^2y + y^3 - 6y^2 + 2x^2 + 12y - 8 = 0$.
6. $ay = (x - b)^{\frac{5}{2}}$.

7. Trace the curve $x^4 + x^3y - 4x^2y + y^3 = 0$, discussed in illustrative Ex. 2.

SUGGESTION: Examine for asymptotes, then put $y = mx$ and find values of x for assumed values of the slope m.

8. Trace the curve of Ex. 1. **9.** Trace the curve of Ex. 6.

MISCELLANEOUS EXERCISES

1. Test for convergence the following series, and determine the interval of convergence:

$$(a)\ 1 + \frac{1}{6}\left(\frac{x}{3}\right) + \frac{1}{11}\frac{1\cdot 4}{1\cdot 2}\left(\frac{x}{3}\right)^2 + \frac{1}{16}\frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{x}{3}\right)^3 + \cdots.$$

$$(b)\ 1 + \frac{x}{e} + \frac{1}{2!}\frac{x^2}{e^2} + \frac{1}{3!}\frac{x^3}{e^3} + \cdots.$$

$$(c)\ 1 + x\cos a + \frac{x^2}{2!}\cos 2a + \frac{x^3}{3!}\cos 3a + \cdots.$$

$$(d)\ \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \cdots.$$

Expand the following functions.

2. $\tan\theta$. **3.** $e^x\cos x$. **4.** $\cos^n x$. **5.** $\dfrac{x}{e^x - 1}$. **6.** $\log\sec^2\dfrac{\theta}{2}$.

7. Show that $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \cdots\right)$.

SUGGESTION: Let $u = (1+x)^{\frac{1}{x}}$, whence $\log u = \dfrac{\log(1+x)}{x}$. Make use of the series already found for $\log(1+x)$ to determine the successive derivatives.

8. From the expansion for $\log(n+h)$ and $\log(n+1)$, derive the *approximate rule of proportional parts*, viz.:

$$\frac{\log(n+h) - \log n}{\log(n+1) - \log n} = \frac{h}{1}.$$

From this rule find log 7.523, knowing that log 7.52 = 2.0176 and log 7.53 = 2.0189.

9. Develop $f(5 \pm h)$ in powers of h, and determine the values of $f(x) = x^2(16 - x)$ for the following values of x: 4.7, 4.8, 4.9, 5, 5.1, 5.2, 5.3.

10. Show by development in series that $e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$, and $e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x$.

11. Develop the function $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$ in series.

SUGGESTION: Multiply by $\sqrt{1-x^2}$ and differentiate. The resulting equation is $(1-x^2)\dfrac{dy}{dx} - xy = 1$. Now assume $y = A + Bx + Cx^2 + \cdots$ and determine the coefficients.

12. Derive the approximate formula $\tan(\theta + m) = \tan\theta + m\sec^2\theta$.

13. The strength of an electric current as shown by a tangent galvanometer is given by $i = C\tan\phi$, where C is a constant and ϕ is the deflection of the needle. Show that an error m in the reading of the angle gives a relative error of $\frac{2m}{\sin 2\phi}$ in the current.

Evaluate the following expressions by the use of series:

14. $\left.\frac{e^x - 1}{x}\right]_{x=0}$. **15.** $\left.\frac{x - \sin x}{x^3}\right]_{x=0}$. **16.** $\left.\frac{e^x - e^{-x}}{\sin x}\right]_{x=0}$.

Calculate the following.

17. Sin 10° and cos 10° to five figures.

18. Logarithms (natural) of 17, 31, 61, correct to four places.

19. $\sqrt[7]{2184}$.

20. Prove that the expansion of an *even* function of x, that is, one for which $f(x) = f(-x)$, will contain only even powers of x, while if $f(x) = -f(-x)$ the expansion will contain only odd powers of x. Illustrate by several functions.

Making use of the known series for e^x, $\log(1+x)$, $\sin x$, etc., derive series for the following functions.

21. $\frac{e^x}{1+x}$. **22.** $e^x\log(1+x)$. **23.** $\sqrt{1 \pm \sin 2x}$. **24.** $\cos^2 x$.

25. By substituting mx for x in Ex. 10, prove *DeMoivre's theorem*, namely,
$$(\cos x + \sqrt{-1}\sin x)^m = \cos mx + \sqrt{-1}\sin mx.$$

26. By means of the exponential series and the identity
$$e^{x\sqrt{-1}}\, e^{y\sqrt{-1}} = e^{x\sqrt{-1}+y\sqrt{-1}}$$
show that $2\sin x\cos x = \sin 2x$.

27. Denoting by c the chord of the half arc, Fig. 101, derive Huygen's approximation to the length of a circular arc, viz.: $s = \frac{8c - C}{3}$.

SUGGESTION. Expand $\sin\frac{\alpha}{2}$ and $\sin\frac{\alpha}{4}$ in series and combine the results.

28. Referring to Fig. 101, deduce an approximate formula for $s - C$ in terms of the chord C and the dimension h.

29. Examine for singular points and asymptotes the curve
$$x^5 - 5ax^2y^2 + y^5 = 0,$$
and trace the curve.

30. Examine for singular points and asymptotes the curve
$$x^4 - 4x^2 - x^2y^2 + y^2 = 0.$$
Trace the curve.

ANSWERS

The answers to some of the problems have been purposely omitted.

Art. 8. Page 11.

1. -26; -14; -110.
3. 1; $\frac{1}{2}\sqrt{2}$; 0; -1.
6. $3-\sqrt{y^2+4}$; $(3-\sqrt{x})^2+4$; $(y^2+4)^2+4$; $3-\sqrt{3-\sqrt{x}}$.
7. $z=-\dfrac{1}{x}$.
12. $\sin(x \pm y)$.
14. $\cos\theta$; $\sin\theta$.
15. $\sec\theta$.
16. $\tan(x-y)$.

Art. 12. Page 17.

2. $x=2$, $x=3$. 4. (*a*) Discontinuous at $x=0$. (*b*) Discontinuous at $x=0$.

Art. 15. Pages 24, 25.

1. $-\frac{3}{5}$.
2. $\frac{1}{2}$.
3. 0.
4. 1.
5. 48.
6. $5a^4$.
11. $-\frac{5}{6}$; 0.
12. 1.

Miscellaneous Exercises. Page 25.

1. (*a*) $(2x-3)\Delta x+(\Delta x)^2$. (*b*) 0.31. 2. (*b*) $3x_1^2$. 12. (*a*) $\frac{1}{2}$. (*b*) $\frac{1}{2}$.

Art. 17. Page 30.

1. $4x^3$.
2. $2x-4$.
3. $-\dfrac{2a}{x^3}$.
4. $\frac{1}{2}x^{-\frac{1}{2}}$.
5. $3x^2-1$.
6. $-\dfrac{1}{(x-1)^2}$.
7. $3(x-a)^2$.
8. gt.
9. $a+gt$.
10. $a-\dfrac{b}{\theta^2}$.
11. $-\dfrac{k}{(v-b)^2}$.

Art. 28. Pages 37, 38.

1. $6x-4$.
2. $12x^2-4x$.
3. $3x^2-10x$.
4. $3x^2-2x-2$.
5. $x^2(6x^3+5x^2-8x-6)$.
6. $6x^2-8x+3$.
7. $\dfrac{6-x}{x^3}$.
8. $-\dfrac{2}{x^3}$.
9. $-\dfrac{ma}{(ax+b)^2}$.
10. $\dfrac{ad-bc}{(cx+d)^2}$.
11. $1+\dfrac{2}{\theta^3}$.
12. $-5\dfrac{(t^2+1)}{(t^2-1)^2}$.
13. $\dfrac{2(2x^2-3x-4)}{(x^2+2)^2}$.
14. $\dfrac{-x^2+12x+19}{(x^2+4x-5)^2}$.

Art. 30. Page 40.

1. $\frac{3}{2}\sqrt{x}$.
2. $2x + x^{-2}$.
3. $\frac{1}{2}\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x^3}}\right)$.
4. $8(2x-5)^3$.
5. $4(x-2)(x^2-4x+3)$.
6. $3x(x^2-a^2)^{\frac{1}{2}}$.
7. $\frac{2x}{3}(1+x^2)^{-\frac{2}{3}}$.
8. $-\frac{4}{3}(4x+3)^{-\frac{4}{3}}$.
9. $\frac{2a^2x}{(x^2-a^2)^{\frac{1}{2}}(x^2+a^2)^{\frac{3}{2}}}$.
10. $-\left(\frac{a}{x^2} + \frac{2b}{x^3} + \frac{3c}{x^4}\right)$.
11. $D_t p = -\frac{B}{t^2} - \frac{2C}{t^3}$.
12. $D_\theta \rho = 2\theta + \frac{k}{2\sqrt{\theta}}$.
14. $x^{p-1}(1-x)^{q-1}(p - px - qx)$.
15. $\frac{x^2(4x^2-15)}{\sqrt{x^2-5}}$.
16. $-\frac{2}{3}\frac{a^2}{x^{\frac{1}{3}}(x^2-a^2)^{\frac{4}{3}}}$.
18. $y = 3\log\frac{x}{c}$.
19. $x = a \text{ arc cos } \frac{a-y}{a} \pm \sqrt{2ay - y^2}$.

Art. 31. Page 42.

1. $\frac{2x}{3}(x^2+5)^{-\frac{2}{3}}$.
2. $\frac{x-1}{\sqrt{x^2-2x+5}}$.
3. $-5x(a^2-x^2)^{\frac{3}{2}}$.
4. $5x(a^2-x^2)^{-\frac{7}{2}}$.
5. $\frac{2}{3}x^{-\frac{2}{3}}(x^{\frac{1}{3}} + a^{\frac{1}{3}})$.
6. $\frac{2x^{\frac{1}{3}}(x^2-2a^2)}{3(x^2-a^2)^{\frac{4}{3}}}$.
7. $(2x - \frac{4}{3}a)(3x^2 - 4ax)^{-\frac{2}{3}}$.
8. $\frac{m+n+2x}{2\sqrt{(x+m)(x+n)}}$.
9. $-\frac{3x^2-2}{3(x^3-2x+5)^{\frac{4}{3}}}$.
10. $\frac{3}{2}(5x^2+6x-12)\sqrt{\frac{x+3}{x^2-4}}$.
11. $\frac{1-\theta^2}{2\theta^{\frac{1}{2}}(1+\theta^2)^{\frac{3}{2}}}$.
12. $-\theta(1+\theta^2)^{-\frac{3}{2}}$.
13. $\frac{(x+\sqrt{x^2-a^2})^{\frac{1}{2}}}{2\sqrt{x^2-a^2}}$.
14. $\frac{x^8}{(x^2+1)^{\frac{3}{2}}}$.
15. $\frac{x(2a^2-x^2)}{(a^2-x^2)^{\frac{3}{2}}}$.

Art. 32. Page 44.

1. $\frac{1}{3(x-4)^{\frac{2}{3}}}$. 2. $\frac{6x}{(1-x^2)^2}$. 3. $\frac{a}{2\sqrt{a\theta+b}}$. 4. $\sqrt{\frac{g}{2s}}$. 6. $\frac{1}{2}(e^y - e^{-y})$.

Art. 33. Pages 44, 45.

1. $6t-1$. 2. $\frac{2t^3-3t}{t-1}$. 3. $\frac{1}{4t^{\frac{3}{2}}}$. 4. $\frac{b-ct}{a}$; $\frac{a}{b-ct}$.

Miscellaneous Exercises. Pages 45, 46.

1. $20x^3 + 9x^2 - 8$.

2. $x(15x^3 + 12x^2 - 12x - 8)$.

3. $\frac{1}{3}x^{-\frac{2}{3}}(4x^{\frac{1}{2}} - a^{\frac{1}{2}})(x^{\frac{1}{2}} - a^{\frac{1}{2}})$.

4. $\dfrac{2x-5}{(x-2)^3}$.

5. $\dfrac{x^3}{\sqrt{1+x^2}}$.

6. $\dfrac{3}{x^4\sqrt{x^2-1}}$.

7. $\dfrac{1}{4\sqrt{a+\sqrt{x}}}$.

8. $\dfrac{nx^{n-1}}{(1+x)^{n+1}}$.

9. $-\dfrac{nx+mx+an}{x^{n+1}(a+x)^{m+1}}$.

10. $\dfrac{n(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$.

11. $\dfrac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}$.

12. $\dfrac{1}{(x+1)\sqrt{x^2-1}}$.

13. $\dfrac{x^2-6x-5}{2(x-3)^{\frac{3}{2}}\sqrt{x^2+5}}$.

14. $\dfrac{-(2+x)\sqrt{1-x^2}}{(1+x)^2}$.

15. $2x + \dfrac{2x^3}{\sqrt{x^4-1}}$.

16. $\dfrac{8x^4 - 9a^2x^2 + 3ax^3}{3(x+a)(x^2-a^2)^{\frac{2}{3}}}$.

17. $2x - \dfrac{2x^2-1}{\sqrt{x^2-1}}$.

18. $4x - \dfrac{2(2x^2+1)}{\sqrt{x^2+1}}$.

19. $\dfrac{2a}{v^3} - \dfrac{c}{(v-b)^2}$.

20. $-\dfrac{p^2}{c+p^2ma}$.

22. arc tan $(\pm\frac{3}{4})$.

23. $\dfrac{10x(2-x^2)}{(x^4-10x^2+10)^{\frac{3}{2}}}$.

24. $\dfrac{a}{b+v}$.

25. $-\dfrac{C}{v^2}$; $-\dfrac{C}{p^2}$.

26. (*b*) 1.0025025.

27. $\dfrac{\mu(1-2\mu x-x^2)}{(x+\mu)^2}$; $\dfrac{1}{\mu}$; $\dfrac{1-3\mu^2}{4\mu}$; $-\mu+\sqrt{1+\mu^2}$.

28. 0; $\pm\frac{4}{3}$; $\mp\frac{3}{4}$.

29. $-\dfrac{1}{s}\sqrt{\dfrac{ac}{2as-2s^2}}$.

30. $\dfrac{x^2}{(ax+b)^2}$.

Art. 37. Page 55.

2. (*a*) $\frac{1}{2}$. (*b*) $\frac{1}{2}$.

3. 0.50925.

Art. 38. Page 56.

1. $3x - 8y = -38$; $8x + 3y = -4$.
2. $9x + 4y = 72$; $4x - 9y = -65$.
3. $2x - y = a$; $x + 2y = 3a$.

4. $3x + 4y = 50$; $4x - 3y = 25$.

5. $8x + 5\sqrt{21}\,y = 100$; $5\sqrt{21}\,x - 8y = \frac{18}{5}\sqrt{21}$.

6. $9x - y = 6$; $x + 9y = 110$.

7. $17x - 4y = 20$; $8x + 34y = 135$.

8. $3y_1^2 y = ax_1(2x - x_1)$; $y - y_1 = -\dfrac{3y_1^2}{2ax_1}(x - x_1)$.

9. $y - y_1 = -\dfrac{y_1^{\frac{1}{3}}}{x_1^{\frac{1}{3}}}(x - x_1)$; $y - y_1 = \dfrac{x_1^{\frac{1}{3}}}{y_1^{\frac{1}{3}}}(x - x_1)$.

10. $yy_1^{n-1} + xx_1^{n-1} = a^n$; $y - y_1 = \left(\dfrac{y_1}{x_1}\right)^{n-1}(x - x_1)$.

11. $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$; $y - y_1 = -\dfrac{a^2 y_1}{b^2 x_1}(x - x_1)$.

Art. 39. Page 58.

1. $\frac{4}{3}\sqrt{13}$; $2\frac{2}{3}$; 6.

3. 4; 4; $4\sqrt{2}$; $4\sqrt{2}$.

4. x_1; $\dfrac{y_1^2}{x_1}$; $\sqrt{x_1^2 + y_1^2}$; $\dfrac{y_1\sqrt{x_1^2 + y_1^2}}{x_1}$.

5. $\frac{4}{3}$; $\frac{3}{4}$; $\frac{5}{3}$; $\frac{5}{4}$.

6. $\dfrac{y_1}{3(x_1^2 - 1)}$; $3y_1(x_1^2 - 1)$; $\dfrac{y_1}{3(x_1^2 - 1)}\sqrt{9x_1^4 - 18x_1^2 + 10}$; $y_1\sqrt{9x_1^4 - 18x_1^2 + 10}$.

7. $\dfrac{3a}{2}$; $\dfrac{2a}{3}$; $\dfrac{a}{2}\sqrt{13}$; $\dfrac{a}{3}\sqrt{13}$.

8. $\dfrac{a}{2}$; $2a$; $\dfrac{a}{2}\sqrt{5}$; $a\sqrt{5}$.

9. $x_1^{\frac{1}{3}}y_1^{\frac{2}{3}}$; $\left(\dfrac{y_1^4}{x_1}\right)^{\frac{1}{3}}$; $a^{\frac{1}{3}}y_1^{\frac{2}{3}}$; $\dfrac{a^{\frac{1}{3}}y_1}{x_1^{\frac{1}{3}}}$.

Art. 41. Page 61.

2. $2a\sqrt{\theta_1}$; $\dfrac{a}{2\theta_1^{\frac{3}{2}}}$.

3. $\dfrac{a}{\sqrt{\theta_1}}\sqrt{1 + \dfrac{1}{4\theta_1^2}}$.

4. $a\theta\sqrt{1 + \theta^2}$; $a\theta^2$; $a\sqrt{1 + \theta^2}$; a.

5. $\dfrac{a\theta^2\sqrt{4 + \theta^2}}{2}$; $\dfrac{a\theta^3}{2}$; $a\theta\sqrt{4 + \theta^2}$; $2a\theta$.

6. $\dfrac{(\theta + 1)\sqrt{(\theta^2 + \theta)^2 + 1}}{\theta}$; $(\theta + 1)^2$; $\dfrac{\sqrt{(\theta^2 + \theta)^2 + 1}}{\theta^2}$; $\dfrac{1}{\theta^2}$.

7. $\dfrac{\rho\sqrt{4\rho^4 + (a + 2b\theta)^2}}{a + 2b\theta}$; $\dfrac{2\rho^3}{a + 2b\theta}$; $\dfrac{\sqrt{4\rho^4 + (a + 2b\theta)^2}}{2\rho}$; $\dfrac{a + 2b\theta}{2\rho}$.

8. θ; $\dfrac{\theta}{2}$; $\theta^2 + \theta$; $2\theta\dfrac{a + b\theta}{a + 2b\theta}$.

Art. 43. Page 64.

1. (*a*) $v = v_0 - gt_1$; $a = -g$.
   (*b*) 235.6 ft./sec.; -32.2 ft./sec.^{2}.
   (*c*) -86.4 ft./sec.

2. $18\frac{102}{161}$ sec.

4. 13 rad./sec.; 2 rad./sec.^{2}.

5. (*a*) 32 rad./sec.; -32 rad./sec.^{2}.
   (*b*) $3\frac{1}{2}$ sec.

6. $\omega = a - 3\,bt^2$; $\alpha = -6\,bt$.

7. $\omega = \dfrac{a}{2\sqrt{t}}$; $\alpha = -\dfrac{a\sqrt{t}}{4\,t^2}$.

8. $\omega = b + 2\,ct$; $\alpha = 2\,c$.

Art. 44. Page 66.

1. 0.54084; 0.23440; 0.23848.

2. -0.00006704; 0.00026816.

Miscellaneous Exercises. Pages 67, 68.

1. (*a*) 0; $-\frac{1}{4}$. (*b*) At $x = 0$ and $x = \frac{4}{3}\,a$. (*c*) At $x = -\frac{2}{3}\,a$ and $x = 2\,a$.

2. $\pm 45^0$. 3. $15\,x + 2\,y = 60$.

9. (*a*) $\rho = k\theta + c$. (*b*) $\theta + \dfrac{k}{\rho} = c$. 11. 0.50925; 0.51040.

Art. 48. Page 77.

1. $dy = -\dfrac{2\,dx}{(x-5)^2}$.

2. $dy = \dfrac{(a^4 - 2\,a^2x^2 - x^4)dx}{(a^2 + x^2)\sqrt{a^4 - x^4}}$.

3. $dy = \dfrac{(2\,x - 5\,a)dx}{15\,x^{\frac{2}{3}}(x-a)^{\frac{6}{5}}}$.

4. $dp = -\dfrac{mcdv}{v^{m+1}}$.

5. $dy = bmn(a + bx^n)^{m-1}x^{n-1}dx$.

6. $dy = -\dfrac{1 + 3\,x}{2\sqrt{1+x}}\,dx$.

7. $dy = -\dfrac{(a^{\frac{1}{3}} - x^{\frac{1}{3}})^{\frac{1}{2}}\,dx}{2\,x^{\frac{2}{3}}}$.

8. $dy = \dfrac{dx}{(2\,ax - x^2)^{\frac{3}{2}}}$.

9. $dy = \dfrac{30\,x^5dx}{\sqrt{2\,x^2 + 1}}$.

10. $dy = -\dfrac{3\sqrt{2\,ax - x^2}}{x^3}\,dx$.

11. $dy = -\dfrac{dx}{x^2\sqrt{x^2 + a^2}}$.

12. $dy = \dfrac{8\,x^3dx}{3(x^2+1)^{\frac{2}{3}}}$.

Art. 49. Page 78.

1. $\dfrac{3\,x^2 - y^2}{2\,xy}$.

2. $\dfrac{b^2x}{a^2y}$.

3. $-3\,xy^{\frac{1}{3}}$.

4. $\dfrac{4\,x^3 - 3\,x^2y + 2\,xy^2 + 2\,y^3}{x^3 - 2\,x^2y - 6\,xy^2}$.

5. $-\dfrac{ax + hy + g}{hx + by + f}$.

6. $\frac{4}{3}$.

7. $-\frac{9}{7}$; $\frac{9}{7}$.

8. $\frac{1}{2}$.

9. (a) Subtan. $= 4\rho^2\sqrt{\theta} - 2\rho\theta$;

Subn. $= \dfrac{\rho}{4\rho\sqrt{\theta} - 2\theta}$.

(b) Subtan. $= 1 - 2\rho\theta$;

Subn. $= \dfrac{\rho^2}{1 - 2\rho\theta}$.

10. (a) $-\dfrac{np}{mv}$; (b) $\dfrac{pv^3 - av + 2ab}{v^3(b - v)}$.

11. $\dfrac{2y(1+x) + 3x^2 + y^2}{2y(1-x) - (1+x)^2}$.

12. $-\dfrac{y^2 + b^2}{2xy}$.

13. $\dfrac{y}{x} \cdot \dfrac{\sqrt{y} - 2\sqrt{x}}{\sqrt{x} - 2\sqrt{y}}$.

Art. 50. Pages 80, 81.

1. $\frac{18}{5}$; $\frac{6}{5}\sqrt{34}$.

2. 12; -9.

3. (a) 6 ft./sec.; (b) 10 ft./sec.

4. (a) $\theta = a\rho$; (b) $\rho^2 = k\theta$.

5. 2π cu. in./sec.

6. $3\frac{3}{4}$ mi./hr.

7. $\dfrac{\pi}{\sqrt{gL}}$.

8. 0.25 per cent approx.

9. $\dfrac{\Delta V}{V} = 3\dfrac{\Delta D}{D}$.

11. $\dfrac{72}{\pi}$ ft./sec.

Miscellaneous Exercises. Pages 81, 82.

1. $dy = -\dfrac{dx}{x^4\sqrt{1-x^2}}$.

2. $dy = -\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{x^2\sqrt{1-x^4}}dx$.

3. $dy = \dfrac{5}{3}\left[\dfrac{6x^3 + 35x^2 - 12x - 56}{x^{\frac{1}{3}}(x^2-4)^{\frac{1}{2}}(x+7)^{\frac{2}{3}}}\right]dx$.

4. $dy = \dfrac{\sqrt{a}\sqrt{a-x} - 2x}{2\sqrt{a^2 - x^2}}dx$.

5. $\dfrac{3x^2 - 10axy}{5ax^2 - 21y^2}$.

6. $\dfrac{2x(1-y^2)^{\frac{3}{2}} - 3y(1-y^2)}{2y(1-y^2)^{\frac{3}{2}} + 3x}$.

7. $-\dfrac{y^2}{x^2}$.

8. $-\dfrac{x}{y}$.

9. $v_x = \dfrac{12y}{\sqrt{100+y^2}}$; $v_y = \dfrac{120}{\sqrt{100+y^2}} \cdot 6\sqrt{2}$; $6\sqrt{2}$.

10. 4 ft./sec.; 3 ft./sec.

11. $\frac{4}{3}\sqrt{11}$; $6\frac{2}{3}$.

12. $17\frac{7}{9}$; $\frac{5}{3}\sqrt{1105}$.

13. $14\sqrt{3}$; 14.

15. $a_y = \dfrac{c^{\frac{2}{3}}k^2}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$, where $k = D_t x$.

16. $y + cx^2 = 0$.

17. $\dfrac{C\tan\frac{1}{2}\beta}{\pi a^2}$ ft./sec.

18. $\dfrac{\sqrt{16.1}}{50}$ ft./sec.^{2}.

20. -2.24 lb./sq. in. per sec.

21. $0.1\,R_0(a + 2b\tau)$.

22. $\dfrac{R_0(e - 2f\tau_1)}{(1 - e\tau_1 + \tau f_1^2)^2}$.

23. (a) $\pm\dfrac{a}{4}\sqrt{6}$; (b) $0.899\,a$.

(c) $y - \dfrac{a}{2}\sqrt{2\sqrt{3} - 3} = -(3 + 2\sqrt{3})\sqrt{2\sqrt{3} - 3}\left(x - \dfrac{a}{2}\right)$.

24. $\dfrac{1}{6\pi}$ in./sec.

Art. 56. Pages 86, 87.

1. $-a \sin ax$.
2. $3 \tan^2 x \sec^2 x$.
3. $2[\cos 2x + \sin 2x]$.
4. $2 \sec^2 x \tan x$.
5. $6 \sin 3x \cos 3x$.
6. $6x \cos^2(a^2 - x^2) \sin(a^2 - x^2)$.
7. $x \cos x + \sin x$.
8. $\dfrac{x}{2}\left(x \sec^2 \dfrac{x}{2} + 4 \tan \dfrac{x}{2}\right)$.
9. $-\dfrac{x}{\sqrt{x^2 - a^2}} \sin \sqrt{x^2 - a^2}$.
10. $-\tan^2 x$.
11. $x^2 \csc 2x (3 - 2x \cot 2x)$.
12. $x \sin x$.
13. $x^2 \cos x$.
14. $2 \sin^2 x$.
15. $3 \cos^3 x$.
16. $\dfrac{2a(2 \sin \theta - \sin^3 \theta)}{\cos^2 \theta}$.
17. $-\dfrac{a \sin 2\theta}{\sqrt{\cos 2\theta}}$.
18. $a(\sin n\theta)^{\frac{1-n}{n}} \cos n\theta$.
19. $a \sec^3 \dfrac{\theta}{3} \tan \dfrac{\theta}{3}$.
20. $-\dfrac{\sin \theta}{(1 - \cos \theta)^2}$.
21. $-\omega(A \sin \omega t + B \cos \omega t)$.
22. $r \sin \theta \left[1 + \dfrac{r \cos \theta}{\sqrt{L^2 - r^2 \sin^2 \theta}}\right]$.
23. $\dfrac{\sin \phi}{1 - \cos \phi}$.
24. $a\phi \cos \phi$; $a\phi \sin \phi$; $\tan \phi$.
25. (a) $\dfrac{\sin^2 \theta}{\cos \theta}$; $\cos \theta$.

    (b) $\dfrac{a(1 - \cos \theta)^2}{\sin \theta}$; $a \sin \theta$.

    (c) $a \sec^2 \dfrac{\theta}{2} \cot \dfrac{\theta}{2}$; $a \sec^2 \dfrac{\theta}{2} \tan \dfrac{\theta}{2}$.
27. $\tan x$; $a^2 \sin x \cos x$.
28. $90°$.
29. $\arctan(2\sqrt{2})$.
30. $\dfrac{\pi}{4} + n\pi$.
31. 0.00027; -0.00011.
32. $\frac{1}{4}\sqrt{2}\, ab$.
33. $-kv_0 \sin kt$.

Art. 60. Page 92.

1. $-\dfrac{1}{\sqrt{a^2 - x^2}}$.
2. $\dfrac{\sqrt{k}}{k + x^2}$.
3. $\dfrac{2a^2}{x\sqrt{x^4 - a^4}}$.
4. $-\dfrac{x}{\sqrt{a^2 - x^2}\sqrt{1 - a^2 + x^2}}$.
5. $\arctan x + \dfrac{x}{1 + x^2}$.
6. $-\dfrac{a}{\rho\sqrt{\rho^2 - a^2}}$.
7. $\dfrac{a}{\rho\sqrt{\rho^2 - a^2}}$.
8. 0.
9. 1.
10. $-\dfrac{k^2 \sec^2 x \tan x}{(2 - k^2 \tan^2 x)\sqrt{1 - k^2 \tan^2 x}}$.
11. $2x\left(\arccos x^2 - \dfrac{x^2}{\sqrt{1 - x^4}}\right)$.
12. $\dfrac{1}{n(m^2 + x^2)}$.
13. $-\dfrac{1}{x^2}\sqrt{a^2 - x^2}$.
14. $\dfrac{1}{x}\sqrt{x^2 - a^2}$.
15. $\dfrac{x^2}{\sqrt{a^2 - x^2}}$.
16. $\arccos x$.

Art. 65. Pages 97, 98.

1. $\dfrac{2x-3}{x^2-3x+5}$.

2. $-\dfrac{2m}{(x^2-m^2)\log a}$.

3. $\dfrac{xa^{\sqrt{x^2-a^2}}\log a}{\sqrt{x^2-a^2}}$.

4. $3x^2e^{x^3}+e^x$.

5. $1+\log x$.

6. $\dfrac{1}{x\log x}$.

7. e^x-e^{-x}.

8. $-\dfrac{4}{(e^x-e^{-x})^2}$.

9. $e^{\sin x}\cos x$.

10. $\dfrac{\sec^2 x}{\tan x}$.

11. $-\dfrac{2}{\sqrt{x^2+1}}$.

12. $\dfrac{1}{\sqrt{x^2-a^2}}$.

13. $\dfrac{1}{x\sqrt{x^2+a^2}}$.

14. $\arctan x$.

15. xe^{ax}.

16. $\dfrac{x}{a+bx^2}$.

17. $\dfrac{1-2x+2x^2-3x^3}{\sqrt{1+x^2}}$.

18. $\dfrac{x(x^3+9x^2-24)}{\sqrt{x^2-4}\,(x+3)^3}$.

19. $\dfrac{x^2+1}{x^2(1-x^2)^{\frac{3}{2}}}$.

20. $-\dfrac{1}{(1+x)\sqrt{1-x^2}}$.

21. $\sqrt{k}(Ae^{\theta\sqrt{k}}-Be^{-\theta\sqrt{k}})$.

22. $e^{-kt}(B-Bkt-Ak)$.

23. $e^{-kt}[(ma-bk)\cos mt-(mb+ak)\sin mt]$.

24. $\dfrac{e^{a\theta}}{a}$; $ae^{a\theta}$; $\dfrac{e^{a\theta}\sqrt{a^2+1}}{a}$; $e^{a\theta}\sqrt{a^2+1}$.

25. $45°$; $\arctan\dfrac{1}{e^2}$.

26. $\dfrac{a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})}{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}$; $\dfrac{a(e^{\frac{2x}{a}}-e^{-\frac{2x}{a}})}{4}$; $\dfrac{a(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2}{4}$.

27. 1.4656.

Miscellaneous Exercises. Pages 98–100.

4. (a) $\dfrac{\pi}{4}+n\pi$; $\pm\dfrac{\pi}{4}$; $\dfrac{3}{4}\pi+n\pi$. (b) $\dfrac{n\pi}{2}$; $\arctan(\pm 2)$; $\dfrac{\pi}{4}+\dfrac{n\pi}{2}$.
(c) $\dfrac{3\pi}{4}+n\pi$; $\arctan(\pm\sqrt{2})$; $\dfrac{\pi}{4}+n\pi$. (d) $x=1$; $\dfrac{\pi}{4}$; nowhere.

5. (a) 0.80902. (b) -0.58779. (c) 1.52786.

6. $\dfrac{\sin\theta}{1-\cos\theta}$; 0.5773; π.

7. $\dfrac{\sqrt{a^2-x^2}}{x}$.

8. $\dfrac{x^2}{\sqrt{2ax+x^2}}$.

9. $\sqrt{a^2-x^2}$.

10. $\dfrac{1}{x^4(1+x^2)}$.

11. $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

12. $-\dfrac{a^{\frac{1}{x}}\sec^2 a^{\frac{1}{x}}\log a}{x^2}$.

13. (a) $\dfrac{a(1+\cos\theta)^2}{\sin\theta}$; $a\sin\theta$. (b) $\dfrac{2\rho}{a}$; $\dfrac{a\rho}{2}$.

(c) $\dfrac{1}{2}\rho\tan\theta$; $a^2\sin 2\theta$. (d) $\rho\cot\dfrac{\theta}{2}$; $\rho\tan\dfrac{\theta}{2}$.

14. (a) $-\dfrac{1+\cos\theta}{\sin\theta}$. (b) $\dfrac{2}{a}$. (c) $\dfrac{\tan\theta}{2}$. (d) $\cot\dfrac{\theta}{2}$.

16. $m\cos\dfrac{\theta}{2}$; $m\sin\dfrac{\theta}{2}$.

17. (a) $-p\left(\dfrac{B}{T^2}+\dfrac{2C}{T^3}\right)$. (b) $\dfrac{pnC}{T(T-C)}$. (c) $p(b\alpha^\theta\log\alpha - c\beta^\theta\log\beta)$.

18. $-ae^{-\lambda t}\{2\pi b\sin 2\pi(bt+c)+\lambda\cos 2\pi(bt+c)\}$.

Art. 68. Page 103.

5. $-\dfrac{a}{2x^2}-\dfrac{b}{x}+C$.

6. $\frac{5}{7}x^7 - x^4 + x^2 + 7x + C$.

7. $-\dfrac{3}{x}+4x-\dfrac{x^3}{3}+\dfrac{2x^5}{5}+C$.

8. $-\dfrac{1}{x}-\dfrac{1}{3x^3}-\dfrac{1}{5x^5}+C$.

9. $\frac{3}{5}x^{\frac{5}{3}} - 2x^{\frac{3}{2}} + C$.

10. $\dfrac{16x^9}{9}-12x^8+\dfrac{216x^7}{7}-36x^6+\dfrac{81x^5}{5}+C$.

11. $\dfrac{(ax+b)^3}{3a}+C$.

12. $\dfrac{(x^3+4)^3}{3}+C$.

13. $\dfrac{(2x^4-5)^4}{32}+C$.

14. $\dfrac{(3x^2-7)^5}{30}+C$.

15. $2\sqrt{x^3-5x+7}+C$.

Art 70. Page 108.

1. $\frac{1}{5}(x+a)^5+C$.

2. $\frac{2}{3}(x^2-a^2)^{\frac{3}{2}}+C$.

3. $\log(3x^2+a^2)+C$.

4. $e^{x^2}+C$.

5. $-\dfrac{1}{\sin\theta}+C$.

6. $-\frac{1}{3}\cos^3\theta+C$.

7. $\frac{1}{2}(\text{arc}\tan x)^2+C$.

8. $\text{arc}\sin mx + C$.

9. $\frac{1}{2}\tan^2\theta+C$.

11. $\dfrac{2}{7a}(ax+b^2)^{\frac{7}{2}}+C$.

12. $\frac{1}{6}\log(2x^3-5)+C$.

13. $\sqrt{a^2+x^2}+C$.

14. $\dfrac{1}{c}\,\text{arc}\sin cx + C$.

15. $\text{arc}\sec ax + C$.

16. $\dfrac{a^{m^2x^2}}{2m^2\log a}+C$.

17. $\frac{1}{2}\,\text{arc}\tan\dfrac{x+1}{2}+C$.

18. $\text{arc}\sin\dfrac{x-4}{5}+C$.

19. $\log(x+3+\sqrt{x^2+6x+10})$.

20. $\dfrac{1}{12}\log\dfrac{1+x}{11-x}$, if $-1<x<11$.

Art. 71. Page 110.

1. $\frac{1}{b^2}[a + bx - a \log(a + bx)]$.
2. $\frac{\sqrt{x^2 - a^2}}{a^2 x}$.
3. $\frac{x}{a^2\sqrt{a^2 - x^2}}$.
4. $\frac{2}{3}(4 + x)\sqrt{x - 2}$.
5. $\log \frac{\sqrt{2x+1} - 1}{\sqrt{2x+1} + 1}$.
7. $\text{arc cos}\,\frac{a - x}{a}$.

Miscellaneous Exercises. Pages 111, 112.

1. $\frac{m}{a-2}\,x^{a-2} + C$.
2. $2\sqrt{3x - x^2} + C$.
3. $\frac{-2}{3\sqrt{x^3 - a^3}} + C$.
4. $\log\sqrt{x^2 - 6x + 1} + C$.
5. $\sqrt{x^2 - 6x + 1} + C$.
6. $\frac{1}{2}e^{x^2} + C$.
7. $-\frac{1}{3}\cos^3\theta + C$.
8. $-\frac{1}{2}\csc^2\theta + C$.
9. $2\,\text{arc sin}\,\frac{x}{2} - 3\sqrt{4 - x^2}$.
10. $-\frac{2}{a\sqrt{ax + b}} + C$.
11. $\frac{1}{9}\log(3x^3 - 5) + C$.
12. $\log\sqrt{x^4 - 5x^2 + 2x - 7} + C$.
13. $\frac{1}{2}\,\text{arc tan}\,\frac{x+3}{2} + C$.
14. $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \log(x+1) + C$.
15. $\log(x^3 + x) - \frac{1}{2}\log(x^2 + 1) + C$.
16. $-\frac{1}{b}\log(a + b\cos x) + C$.
17. $-e^{\cos x} + C$.
18. $\text{arc tan}\,x - \log\sqrt{1 + x^2} + C$.
19. $\log\frac{2x+1}{2x+1} + C$.
20. $\text{arc sin}\,\frac{2x+3}{\sqrt{57}} + C$.
21. $\log[x + 3 + \sqrt{x^2 + 6x + 1}] + C$.
22. $\frac{1}{2\sqrt{6}}\log\frac{z - \sqrt{6}}{z + \sqrt{6}} + C$.
23. $-\frac{1}{4\sqrt{3}}\log\frac{\sqrt{3}\,\theta - 2}{\sqrt{3}\,\theta + 2} + C$.
24. $\text{arc sin}\,(\log x) + C$.
25. $\log[s + a + \sqrt{s^2 + 2as}] + C$.
26. $\log c[u + m + \sqrt{(u + m)^2 - n^2}] + C$.
27. $\frac{1}{4}\,\text{arc sin}\,\frac{x^4}{a^4} + C$.
28. $\text{arc sin}\,x + \sqrt{1 - x^2} + C$.
29. $\log\frac{1}{1 + e^{-x}} + C$.
30. $2\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right) + C$.
33. $y = \log\sqrt{x^2 + 1} + 2$.
34. $y = \frac{b}{2}(x^2 - 1) + a\left(1 - \frac{1}{x}\right)$.
35. $s = \frac{v_0}{k}\sin kt$.

Art. 73. Page 114.

1. $y = 3x + C$; $y = \frac{1}{6}x^3 + C$; $y = \frac{1}{2}mx^2 + C$; $y = \frac{1}{x} + C$; $y = \frac{1}{2}ax^2 - bx + C$;
$$y = -\frac{1}{2ax^2} + C.$$

2. $y = x^2 + C$.
3. $y = bx + C$.
4. $y^3 = mx + C$.
5. $\theta = k\rho + C$.
6. $m\theta = \rho + k$.
7. $y = \frac{3}{2}x^2 + 5x - 13$.
8. $y = Ce^{ax}$.
9. $\log y = \frac{x}{a} + C$.
10. $\rho = e^{a\theta}$.

Art. 76. Page 118.

1. (a) $v = v_0 + mt - \frac{1}{3}nt^3$;
$s = s_0 + v_0 t + \frac{1}{2}mt^2 - \frac{1}{12}nt^4$.
(b) $v = v_0 - mk \sin kt$;
$s = s_0 + v_0 t + m \cos kt$.
(c) $v = \frac{v_0}{2}(e^{kt} + e^{-kt})$; $s = \frac{a}{k^2}$.
2. $a = 10 - 2t_1$; $s = 5t_1^2 - \frac{1}{3}t_1^3$.
3. 10 sec.; $166\frac{2}{3}$ ft.; -10 ft./sec.^{2}.
4. 8 sec.; 15.27 rev.
7. $y = -\frac{gx^2}{2v_0^2}$.
8. 60 ft./sec.

Miscellaneous Exercises. Pages 124, 125.

1. $\frac{\sqrt{3}}{6} \text{ arc tan } \frac{x^2}{\sqrt{3}} + C$.
2. $\frac{1}{3}x^3 + x + \frac{1}{2}\log\frac{x-1}{x+1} + C$.
3. $\frac{2}{3}\sqrt{e^x + 1}(e^x - 2) + C$.
4. arc tan $e^x + C$.
5. $\frac{3}{\sqrt{7}} \text{ arc sec } \frac{2x}{\sqrt{7}} + C$.
6. $\frac{1}{b} \text{ arc sec } \frac{x-a}{b} + C$.
7. (a) $y = \frac{3}{2}x^2 + 2x + C$.
(b) $y = \frac{9}{4}x^{\frac{4}{3}} + Cx + C'$.
(c) $y = \frac{1}{m}\sin mx + C$.
(d) $y = \frac{1}{c}e^{cx} + C'$.
8. $y = 4x - \frac{1}{3}x^3 - 5$.
9. $Q = a\tau + \frac{1}{2}b\tau^2 + \frac{1}{3}c\tau^3$.
13. $\rho = ce^{n\theta}$.
14. (a) $\frac{w(x-L)^3 + wL^3}{6}$.
(b) $\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}$.
(c) $\frac{wL^2x^2}{16} - \frac{wx^4}{24}$.
(d) $\frac{3wLx^2}{16} - \frac{wx^3}{6} - \frac{wL^3}{48}$.

Art 82. Page 129.

1. $6x$.
2. $x^x\left[\frac{1}{x} + (\log x + 1)^2\right]$.
3. $e^{ax}[(a^2 - 1)\cos x - 2a\sin x]$.
4. $-\frac{x}{(1-x^2)^{\frac{3}{2}}}$.
5. $-\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.

6. $\dfrac{2(a^2 - x^2)}{(a^2 + x^2)^2}$.

7. $e^x(x + 3)$.

8. $\dfrac{2}{(x-3)^3}$.

9. $-(x \cos x + 3 \sin x)$.

11. $a^x(\log a)^n$.

12. $\dfrac{(-1)^n n!}{x^{n+1}}$.

13. $\dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

14. (a) $-\dfrac{4p^2}{y^3}$. (b) $-\dfrac{b^4}{a^2y^3}$.

15. $\dfrac{12(x^7y + 7x^6y^2 + 23x^5y^3 + 38x^4y^4 + 23x^3y^5 + 7x^2y^6 + xy^7)}{(x^3 + 6x^2y + 3xy^2)^3}$.

16. $\dfrac{(a^2-1)(y^2 - 2axy + x^2)}{(y-ax)^3}$.

17. $\dfrac{1}{x^2}$.

18. $\dfrac{2 + \cos\theta}{(1-\cos\theta)^2}$.

19. $2 \csc^2\theta \cot\theta$.

Art. 83. Pages 131, 132.

1. $D^{-4}(\sin ax) = \dfrac{1}{a^4}\sin ax + \dfrac{c_1}{6}x^3 + \dfrac{c_2}{2}x^2 + c_3x + c_4$.

2. $D^{-4} = \dfrac{1}{360}x^6 - \dfrac{1}{24}x^4 + \dfrac{1}{6}c_1x^3 + \dfrac{1}{2}c_2x^2 + c_3x + c_4$.

3. $D^{-4} = -\frac{1}{6}\log x + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$.

4. $y = \dfrac{kx^3}{6} - \dfrac{m^3k - 6n}{6m}x$.

5. $EIy = \frac{1}{2}Mx^2 + \frac{1}{6}Rx^3 - \frac{1}{24}wx^4 + C_1x + C_2$.
$C_2 = 0$; $C_1 = -\frac{1}{2}Ml - \frac{1}{6}Rl^2 + \frac{1}{24}wl^3$.

Art 84. Page 136.

1. Max. for $x = 0$; min. for $x = 6\frac{2}{3}$.
2. Max. for $x=0$; min. for $x=\pm\sqrt{3}$.
3. Max. for $x = -\frac{1}{3}\sqrt{3}$; min. for $x = \frac{1}{3}\sqrt{3}$.
4. Max. for $x = 0$; min. for $x = \frac{2}{3}$.
5. Min. for $x = 2$.
6. Min. for $x = 1$.
7. Min. for $x = \sqrt[3]{2}$.
8. Min. for $x = 0$.
9. Max. for $x = 1$.
10. Max. for $x = \dfrac{\pi}{4}$.
11. Min. for $x = \frac{3}{4}\pi$, $\frac{7}{4}\pi$, etc.
12. Min. for $x = 2$.
13. Min. for $x = -\dfrac{2b}{3a}$.
14. Min. for $x = -\frac{2}{3}$.
15. Min. for $x = -\frac{3}{2}$.
16. Max. for $x = e$.
17. Min. for $x = 3$ and for $x = -2$.
Max. for $x = -\frac{3}{4}$.
18. Max. for $x = 0$.

Art. 85. Pages 138, 139.

1. $\dfrac{a}{2}$ and $\dfrac{a}{2}$.

2. $45°$; max. range $= \dfrac{v_0^2}{g}$.

3. $-\mu + \sqrt{1 + \mu^2}$.

4. $\dfrac{a}{6}$.

5. 8 in.

6. Length = diameter.

7. $\sqrt{15}$.

8. $4\sqrt{5}$.

10. Altitude $= 2 \times$ diam. of sphere.
12. Base $=$ altitude $= r\sqrt{2}$.
13. $a\sqrt{2}$, $b\sqrt{2}$.
14. Base $= a\sqrt{3}$, altitude $= \frac{3}{2}a$.
15. Altitude $= \frac{1}{3} \times$ altitude of cone.
Radius of base $= \frac{2}{3} \times$ radius of base of cone.
16. Altitude $= \frac{4}{3}a$; radius of base $= \frac{2}{3}a\sqrt{2}$.
17. Radius of base $= \dfrac{1}{2^{\frac{1}{3}}}\sqrt[3]{\dfrac{3V}{\pi}}$;
altitude $= \sqrt[3]{\dfrac{6V}{\pi}}$.
18. $\dfrac{c}{2}$.
19. $\frac{25}{12}$.
20. Breadth $= \dfrac{14\sqrt{3}}{3}$;
depth $= \dfrac{14\sqrt{6}}{3}$.
21. Breadth $= 6$; depth $= 6\sqrt{3}$.

Miscellaneous Exercises. Pages 140, 141.

1. $\sin x(2 - x^2) + 4x\cos x$.
2. $2e^x\cos\left(x + \dfrac{\pi}{4}\right)$.
3. $\dfrac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}y^{\frac{1}{3}}}$.
4. $\dfrac{3a^4x}{(x^2 - a^2)^{\frac{5}{2}}}$.
8. (a) $4e^x(\sin x - \cos x)$; (b) $\dfrac{6}{x}$.
9. $y = \dfrac{wx^2(l - x)^2}{24\,EI}$.
10. $y = ax^2 + c_1x + c_2$;
$y = ax^2 + c_1x$;
$y = ax^2 + 5\,ax + c_2$.
11. (a) $D^{-3} = -\dfrac{a}{2}\log x + \dfrac{1}{24}x^4 + \dfrac{5}{6}x^3 + \dfrac{1}{2}c_1x^2 + c_2x + c_3$.

(b) $D^{-3} = \dfrac{1}{a^3}(e^{ax} + e^{-ax}) + \dfrac{1}{2}c_1x^2 + c_2x + c_3$.

(c) $D^{-3} = \dfrac{1}{k^3}\cos(kt + e) + \dfrac{1}{2}c_1t^2 + c_2t + c_3$.

12. (a) $\alpha^2c_1e^{-\alpha t} + \beta^2c_2e^{-\beta t}$.
(b) $\alpha e^{-\alpha t}[\alpha(c_1 + c_2t) - 2\,c_2]$.
13. $e^{-\alpha t}[(\alpha^2 - \beta^2)(c_1\cos\beta t + c_2\sin\beta t) + 2\,\alpha\beta(c_1\sin\beta t - c_2\cos\beta t)]$.
15. $\sqrt{\dfrac{gh}{\sqrt{1 - c^2}} - gh}$; $1 - \sqrt{1 - c^2}$.
16. (a) Max. for $x = 3$; max. value $= 2$.
Min. for $x = -1$; min. value $= \frac{2}{3}$.
(b) Max. for $\theta = \frac{1}{3}\pi$; max. value $= \frac{3}{4}\sqrt{3}$.
Min. for $\theta = \frac{5}{3}\pi$; min. value $= -\frac{3}{4}\sqrt{3}$.
17. $-\dfrac{1}{a(1 - \cos\theta)^2}$.
18. $\dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$.
19. $r\omega_0^2\left[\cos\theta + \dfrac{rL^2 + r^3\sin^4\theta}{(L^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right]$.

Art. 86. Page 143.

1. Concave up if $x > \frac{2}{3}$; concave down if $x < \frac{2}{3}$.
2. Concave up if $-\frac{\pi}{2} + 2n\pi < x < \frac{\pi}{2} + 2n\pi$;

   concave down if $\frac{\pi}{2} + 2n\pi < x < \frac{3}{2}\pi + 2n\pi$.
3. Concave up if $x < -1$; concave down if $x > -1$.
4. Concave down at every point.
5. Concave up at every point.
8. (*a*) Concave up at every point. (*b*) Concave down at every point.
9. Concave up at every point.

Art. 87. Pages 144, 145.

1. $x = \frac{1}{2}$.
2. No point of inflexion.
3. No point of inflexion.
4. Points of inflexion for $x = n\pi$.
5. Points of inflexion for $x = \frac{n\pi}{2}$, n odd.
6. $x = \log 2$.
7. $x = 0$.
8. $x = 0$.
9. $x = \pm\frac{1}{2}\sqrt{2}$
10. $x = \pm 3, 0$
11. $x = \pm\sqrt{\frac{1}{3b}}$.
13. $y = Cx^3 + x + 3$.

Art. 88. Pages 148, 149.

1. $y = x - \frac{1}{3}b$.
2. $x = a$, $y = \pm(x + a)$.
3. $x = 2a$.
4. $y = \pm\frac{b}{a}x$.
5. $x = 2$.
6. $3x + 3y = 2$.
7. $x = \pm c$; $y = \pm c$.
8. $x = 1$.
9. $x = a$; $y = x + \frac{a}{2}$; $y = -x - \frac{a}{2}$.
10. $y = 0$.
11. $y = x + 2$.
12. $x = 0$; $y = 0$.
13. $y = x + \frac{1}{3}$.
14. $x = 0$; $y = 0$; $x + y = 0$.
15. $x = -1$.
16. No asymptotes.

Art. 89. Pages 151, 152.

1. Double point and cusp at (0, 0); tangents $y^2 = 0$, $y = x$.
2. Cusp at (0, 0); tangents $y^2 = 0$.
3. Tacnode at (0, 0); tangents $y^2 = 0$.
4. Tacnode at (0, 0); tangents $y^2 = 0$.
5. Double point at origin; tangents $y = \pm x$.
6. Double point at origin; tangents $y = 0$, $x = 0$.
11. Double point at $x = a$, $y = b$; slope of tangent $= \pm\sqrt{a}$.
12. Double point at (0, 0); tangents $y = \pm\sqrt{a}x$.

Art. 92. Page 160.

1. $-\frac{(y^2+a^2)^{\frac{3}{2}}}{a^2}$.

2. $\frac{(9a^2y^4+4x^2)^{\frac{3}{2}}}{3ay(6ay^3-8x^2)}$.

3. $-\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

4. $-\frac{[(x+a)^2+1]^{\frac{3}{2}}}{x+a}$.

5. $\sec x$.

6. $3(axy)^{\frac{1}{3}}$.

7. $\frac{y^2}{a}$.

8. $-\frac{(a^4y^2b+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

9. $\frac{(4y^4+x)^{\frac{3}{2}}\sqrt{x}}{2y^4-4x^{\frac{3}{2}}y}$.

10. $\frac{a\sqrt{x(8a-3x)^3}}{3(2a-x)^2}$.

11. $\frac{(x+y)^{\frac{3}{2}}}{c^{\frac{1}{2}}}$.

12. $-15\sqrt{3}$; $(20, -10\sqrt{2})$.

13. $\frac{(109)^{\frac{3}{2}}}{60}$; $(21\frac{1}{6}, 15\frac{9}{20})$.

14. -1; $(0, 0)$.

15. $-\frac{1}{2ak}$.

16. $\frac{2a}{[1+(2ax+b)^2]^{\frac{3}{2}}}$.

17. $x=\frac{1}{2}\log\frac{1}{2}=-0.1534$.

Art. 93. Page 162.

1. $\frac{1}{ab}(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}$.

2. $3a\sin\theta\cos\theta$.

3. $\frac{(\rho^2+a^2)^{\frac{3}{2}}}{\rho^2+2a^2}$.

4. $\frac{a}{\sqrt{2}}$.

5. $\frac{3}{4}a\sin^2\frac{\theta}{3}$.

6. $\rho\sqrt{1+a^2}$.

7. $\frac{8}{3}a$.

8. $\frac{a}{3}$.

Art. 96. Pages 166, 167.

1. The origin. 2. $4(m-2p)^3=27pn^2$. 3. $(am)^{\frac{2}{3}}+(bn)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

4. $m=\frac{a^2-b^2}{a}\cos^3\theta$, $n=-\frac{a^2-b^2}{b}\sin^3\theta$.

5. $m=a\cos\theta$, $n=a\sin\theta$, a circle.

Miscellaneous Exercises. Pages 167, 168.

1. $-\frac{(65)^{\frac{3}{2}}}{16}$; $\left(\frac{89}{4}, -\frac{343}{16}\right)$.

2. $-\frac{(785)^{\frac{3}{2}}}{36}$; $\left(\frac{5477}{9}, -\frac{929}{36}\right)$.

3. -1; $(0, 0)$.

4. $\sqrt{2}$; $(-1, 1)$.

11. Double point at $(a, 0)$; tangents $y=\pm(x-a)$.

12. Conjugate point at $(0, 0)$.

13. Double point at $(0, 0)$; tangents $y = 0$, $y = \frac{1}{2}x$.

14. (a) $y = x + \frac{4}{3}$; (b) $y = \pm\frac{b}{a}x$. 15. $-8a\sin\frac{3}{2}\theta$.

16. $\pm x = a\log\dfrac{a+\sqrt{a^2-y^2}}{y} - \sqrt{a^2-y^2}$.

18. (a) $\rho = a\sec\theta$; (b) the initial line $\theta = 0$.
(c) four asymptotes, $\rho = \frac{a}{2}\sec\left(\frac{\pi}{4}\pm\theta\right)$, $\rho = \frac{a}{2}\sec\left(\frac{3\pi}{4}\pm\theta\right)$.

20. $4096a^3m + 1152a^2n^2 + 27n^4 = 0$.

Art. 97. Page 170.

9. $\log\sqrt{3}$.
10. $\sqrt{2}-1$.
11. $\frac{1}{2}$.
12. 1.
13. 2.
14. $r\arcsin\frac{r}{a}$.
15. $\frac{3}{4}(a^{\frac{2}{3}}-1)^2$.
16. 2.
17. $\frac{\pi}{4}$.
18. $\dfrac{10\sqrt{10}-1}{54}$.

Art. 99. Pages 172, 173.

1. $\frac{1}{2}\arctan 4$.
2. $\frac{\pi}{6}$.
3. $\frac{1}{3}$.
4. $\frac{1}{2}(e^4-1)$.
5. $\arctan e - \frac{\pi}{4}$.
6. $\frac{1}{2}$.
7. $\frac{1}{3}\log^3 2$.
8. $\dfrac{1}{a^2\sqrt{2}}$.

Art. 103. Page 181.

1. $\frac{\pi}{2a}$. 2. $\frac{1}{a}$. 3. 2. 4. $\frac{\pi}{2}$. 8. $\frac{3}{2}(1+\sqrt[3]{15})$.

Miscellaneous Exercises. Pages 182, 183.

1. $\frac{2}{3}\sqrt{2gs_1^3}$.
2. $\frac{2}{3}a^3$.
3. $\frac{2}{15}$.
5. (a) $\frac{1}{2}(\cos\alpha+\cos\beta)$; (b) $\frac{2}{3}a$.
6. $e^4 - e$.
9. $2\sqrt{2}$.
10. $\frac{\pi}{2}$.
11. ∞.

Art. 105. Pages 187, 188.

1. 78.
2. $106\frac{2}{3}$.
3. 10.
4. 32.
6. $a^2(e^{\frac{m}{a}} - e^{\frac{-m}{a}})$.
7. $\dfrac{mC^{\frac{1}{m}}}{m-1}\left(b^{\frac{m-1}{m}} - a^{\frac{m-1}{m}}\right)$; $C\log\frac{b}{a}$.
8. $\dfrac{4\pi^3a^2}{3}$.
9. a^2
10. $4\pi^2a^2$.
11. $\frac{4}{3}ab$.
12. a^2.
14. $\frac{a^2}{n}$.

Art. 106. Page 190.

1. (a) $\frac{1}{3}\pi n^2 m$; (b) $\frac{1}{3}\pi m^2 n$.
2. 27π.
3. 144π.
4. π.
5. $\frac{1}{2}\pi a^2(e^c + 2c - e^{-c})$.
6. $\frac{1}{15}\pi a^3$.
7. $\frac{8}{3}\pi a^3(3\log 2 - 2)$.
8. $\frac{\pi}{2}$.

Art. 107. Page 192.

1. 108π.
2. $\frac{8}{3}\pi abc$.
5. $a^2h(\pi - \frac{4}{3})$.
6. $349\frac{1}{3}$ cu. in.
7. 240.

Art. 108. Page 194.

1. $6a$.
2. $\frac{8a}{27}\left[\left(1+\frac{9m}{4a}\right)^{\frac{3}{2}} - 1\right]$.
3. $\frac{a}{2}\left(e^{\frac{x_1}{a}} - e^{\frac{-x_1}{a}}\right)$.
4. $2\pi a$.
5. $\frac{1}{2}\pi^2 a$.

Art. 109. Page 196.

1. $2\pi a$.
2. $\frac{\sqrt{1+a^2}}{a}\left(e^{\frac{\pi a}{2}} - 1\right)$; $\frac{\sqrt{1+a^2}}{a}\left(e^{\pi a} - e^{\frac{\pi a}{2}}\right)$.
3. $\frac{\sqrt{1+a^2}}{a}$.
4. a.

Art. 110. Page 198.

1. $\frac{32\pi}{3}(2\sqrt{2} - 1)$; $\frac{32\pi}{3}(5\sqrt{5} - 2\sqrt{2})$.
2. $\pi n\sqrt{m^2+n^2}$.
3. $\frac{2\pi}{27}[(37)^{\frac{3}{2}} - 1]$.
4. $\frac{\pi a^2}{4}(e^2 - e^{-2} + 4)$.

Art. 111. Page 200.

1. $\frac{8}{3}\sqrt{5}$.
2. $-5\frac{2}{3}$.
3. (a) $\frac{2}{\pi}$; (b) 0.
4. $\frac{n^2}{6}$.
5. $p_1\frac{x_1}{x_2 - x_1}\log\frac{x_2}{x_1}$.
6. $\frac{2a}{\pi}$.

Art. 113. Page 204.

1. $4303\frac{1}{3}$ in.-lb.
2. $Mh + \frac{mh^2}{2}$.
4. $W = \frac{k}{3}(s_2^3 - s_1^3)$.
5. 83,219 ft.-lb.
6. 65,684 ft.-lb.
7. 3,273,000 in.-lb.

Miscellaneous Exercises. Pages 204–206.

1. (*a*) 372; (*b*) 48; (*c*) $e-1$.
2. $\dfrac{\pi a^2}{8}$.
3. $\log 2$.
4. $\dfrac{a^2}{6}$.
5. (*a*) $\frac{32}{35}\pi$; (*b*) $\frac{4}{5}\pi$.
6. 128 cu. in.
8. $\frac{2}{3}a^3$.
9. a^2.
10. 144; $\dfrac{(73)^{\frac{3}{2}}-27}{16}$.
11. $\frac{2}{3}mn^2$.
12. $\dfrac{2\,i_0}{\pi}$.
13. $\dfrac{2\,p_0}{\pi}$.
14. $C\log\dfrac{v_2-b}{v_1-b}-\dfrac{a(v_2-v_1)}{v_1v_2}$.
15. $\dfrac{k(s_2^{n+1}-s_1^{n+1})}{n+1}$.
16. 56,530 ft.-lb.

Art. 114. Page 209.

1. $e^x(x^2-2x+2)+C$.
2. $\dfrac{x^{m+1}}{m+1}\log x-\dfrac{x^{m+1}}{(m+1)^2}+C$.
3. $\theta\operatorname{arc\,sin}\theta+\sqrt{1-\theta^2}+C$.
4. $\theta\operatorname{arc\,cot}\theta+\log\sqrt{1+\theta^2}+C$.
5. $\frac{1}{2}(\theta-\sin\theta\cos\theta)+C$.
6. $\dfrac{ae^{ax}\sin bx-be^{ax}\cos bx}{a^2+b^2}+C$
7. $x\log x-x+C$.
8. $\sin\theta\,(\log\sin\theta-1)+C$.
9. $\dfrac{x^3}{3}\operatorname{arc\,tan}x-\dfrac{x^2}{6}+\dfrac{\log(x^2+1)}{6}+C$.
10. $\dfrac{ae^{\frac{x}{a}}}{2}\left(\sin\dfrac{x}{a}+\cos\dfrac{x}{a}\right)+C$.
11. $\cos x+x\sin x+C$.
12. $\dfrac{1}{2}\left[\tan\theta\sec\theta+\log\tan\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)\right]+C$.
13. $\dfrac{a^2}{2}\left[\sqrt{2}+\log(1+\sqrt{2})\right]$.
14. $\dfrac{2\,a^5}{15}$.
15. $\dfrac{x^4}{4}\left[(\log x)^2-\frac{1}{2}\log x+\frac{1}{8}\right]+C$.
16. $\frac{1}{3}$.

Art. 115. Page 211.

1. $\log\dfrac{(x-3)^5}{(x-2)^4}+C$.
2. $\log\sqrt{(x+5)^5(x-3)}+C$.
3. $\dfrac{\log x^{\frac{1}{3}}(x-4)^4}{(x-3)^{\frac{13}{3}}}+C$.
4. $\log\dfrac{(x+2)^5}{(x+1)^3}+C$.
5. $\log\sqrt[5]{\dfrac{3x+1}{x+2}}+C$.

6. $\frac{1}{2}x^2 - x + \frac{4}{5}\log x + \frac{18}{15}\log(x-5) - \frac{3}{3}\log(x+1) + C.$

7. $\log\left[\frac{(x+m)(x-n)}{x}\right] + C.$

8. $\log\left[x^2(x-2)^{\frac{5}{2}}(x+2)^{\frac{9}{2}}\right] + C.$

Art. 115. Page 212.

1. $\log\frac{x-2}{x} + \frac{1}{x-2} + C.$

2. $\log\sqrt[9]{\left(\frac{x+3}{x}\right)^2} - \frac{11}{3(x+3)} + C.$

3. $\frac{9-4x}{2(x-2)^2} + C.$

4. $\log(x+3) + \frac{4}{x+3} + C.$

5. $\log\frac{y-3}{y} - \frac{1}{y} + C.$

6. $\log\frac{(x-1)^8}{x^2} - \frac{1}{x} + \frac{5}{x-1} + C.$

7. $\log\frac{x-1}{x+1} - \frac{1}{x-1} + C.$

8. $\frac{1}{4}\log\frac{x-1}{x+1} - \frac{1}{2(x+1)} + C.$

9. $\log x - \frac{3}{x} + C.$

10. $m\log(m+x) + \frac{2m^2}{m+x} - \frac{m^3}{2(m+x)^2} + C.$

Art. 115. Page 213.

1. $\log\frac{(x-1)^3(x+1)^2}{(x^2+1)^{\frac{5}{2}}} + 4\arctan x + C.$

2. $\frac{1}{4}\log\frac{x-1}{x+1} - \frac{1}{2}\arctan x + C.$

3. $\frac{5\arctan x}{2} - \frac{x}{2(1+x^2)} + \frac{1}{x} + C.$

4. $\frac{1}{4}\log\frac{x^2+1}{x^2+3} + C.$

5. $\frac{1}{6}\log\frac{x^2+1}{x^2+4} + C.$

6. $-\frac{z}{z^2+1}.$

7. $\frac{3}{8}\log\frac{y^2+3}{y-1} + \frac{5}{8}\log(y+1) + \frac{1}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}} + C.$

8. $\frac{1}{a^2+b^2}\left[\log\frac{x+a}{\sqrt{x^2+b^2}} + \frac{a}{b}\arctan\frac{x}{b}\right] + C.$

9. $-\frac{1}{3}\left[\frac{1}{x} + \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}\right] + C.$

10. $\log\frac{\sqrt[4]{x^2+1}}{\sqrt{x+1}} + \frac{\arctan x}{2} + C.$

11. $\log\frac{(1+x+x^2)^{\frac{3}{2}}}{(1+x)^3} - \frac{1}{1+x} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$

12. $\log\frac{x}{\sqrt{1+2x^2}} + C.$

13. $\log\sqrt{x^2+1} + \frac{1}{2(x^2+1)} + C.$

Art. 116. Pages 217, 218.

1. $3\left[\frac{x^{\frac{5}{3}}}{5} - \frac{x^{\frac{4}{3}}}{4} + \frac{x}{3} - \frac{x^{\frac{2}{3}}}{2} + x^{\frac{1}{3}} - \log(x^{\frac{1}{3}}+1)\right] + C.$

2. $-\left[\frac{4x^{\frac{5}{6}}}{5} + \frac{4x^{\frac{2}{3}}}{3} + \frac{64x^{\frac{1}{2}}}{27} + \frac{128x^{\frac{1}{3}}}{27} + \frac{1024x^{\frac{1}{6}}}{81} + \frac{4096}{243}\log(3x^{\frac{1}{6}}-4)\right] + C.$

3. $\frac{2}{m^3}\left[\frac{(mx+b)^{\frac{7}{2}}}{7} - \frac{2b(mx+b)^{\frac{5}{2}}}{5} + \frac{b^2(mx+b)^{\frac{3}{2}}}{3}\right] + C.$

4. $2 \operatorname{arc} \tan \sqrt{\frac{x-1}{3-x}} + C.$

5. $6 \operatorname{arc} \tan \sqrt{\frac{x-2}{5-x}} + C.$

6. $\log[x + \frac{5}{2} + \sqrt{x^2+5x-3}] + C.$

7. $\log[x - \frac{7}{2} + \sqrt{x^2-7x+4}] + C.$

8. $\sqrt{\frac{2}{3}} \operatorname{arc} \tan \sqrt{\frac{3(x-2)}{2(3-x)}} + C.$

9. $2\left\{\frac{(x-1)^{\frac{7}{2}}}{7} + \frac{3(x-1)^{\frac{5}{2}}}{5} + (x-1)^{\frac{3}{2}} + (x-1)^{\frac{1}{2}}\right\} + C.$

10. $3\sqrt{x^2-3} - \log[x+\sqrt{x^2-3}] + C.$

11. $\log[x+1+\sqrt{x^2+2x+5}] + C.$

12. $\frac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + C.$

13. $\frac{x^2+2}{\sqrt{x^2+1}} + C.$

14. $-\frac{2\sqrt{2x-x^2}}{x} - \operatorname{arc} \operatorname{vers} x + C.$

15. $\log[x+\frac{1}{2}+\sqrt{x^2+x}] + C.$

16. $-\frac{\sqrt{a^2-x^2}}{a^2x} + C.$

17. $\frac{x}{\sqrt{a^2-x^2}} - \operatorname{arc} \sin \frac{x}{a} + C.$

18. $-\frac{x}{\sqrt{x^2-a^2}} + \log(x+\sqrt{x^2-a^2}) + C.$

19. $\frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.$

20. $\frac{x}{8}(2x^2-5a^2)\sqrt{x^2-a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2-a^2}) + C.$

21. $\frac{3}{8} \operatorname{arc} \sin x - \frac{2x^3+3x}{8}\sqrt{1-x^2} + C.$

22. $\frac{1}{3}(x^2-2)\sqrt{1+x^2} + C.$

23. $\frac{\sqrt{x^2-1}}{x} - \frac{(x^2-1)^{\frac{3}{2}}}{3x^3} + C.$

24. $\frac{\pi a^2}{4}.$

25. $\frac{11\sqrt{15}}{64}.$

Art 117. Page 221.

1. $\frac{1}{6}\sec^6 x + C.$

2. $-\cot x - \frac{1}{3}\cot^3 x + C.$

3. $-\frac{1}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C.$

4. $-\frac{1}{3}\cot^3 x + C.$

5. $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$

6. $-\frac{1}{2}\cot^2 x + \log \tan x] + C.$

7. $\frac{1}{2}\tan^2 x - \log \sec x + C.$

8. $\frac{1}{6}\tan^6 x + C.$

9. $-\sin x - \csc x + C.$

10. $\frac{2}{3}\tan^{\frac{3}{2}} x + C.$

11. $\frac{3}{4}\sin^{\frac{4}{3}} x - \frac{3}{10}\sin^{\frac{10}{3}} x + C.$

12. $\tan x - \cot x + C.$

Miscellaneous Examples. Pages 228–230.

1. $-\dfrac{1}{\sqrt{1+x^2}}\left(\dfrac{1}{x}+2x\right)+C.$

2. $\dfrac{1}{\sqrt{2}}\operatorname{arc\,sin}\dfrac{4x-3}{\sqrt{41}}+C.$

3. $\frac{8}{13}x^{\frac{13}{12}}-\frac{1}{5}x^{\frac{5}{4}}+C.$

4. $\frac{4}{3}(\sqrt{x}+a)^{\frac{3}{2}}-4a(\sqrt{x}+a)^{\frac{1}{2}}+C.$

5. $\dfrac{(x^2-1)\sqrt{2x^2+1}}{6}+C.$

6. $(\theta+\epsilon)\cos\epsilon+\sin\epsilon\log\sec(\theta+\epsilon)+C.$

7. (a) $\dfrac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\dfrac{3a^4}{8}\operatorname{arc\,sin}\dfrac{x}{a}+C.$

(b) $\dfrac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\dfrac{a^4}{8}\operatorname{arc\,sin}\dfrac{x}{a}+C.$

9. (a) $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\log\sec x+C.$ (b) $\frac{1}{3}\tan^3\theta-\tan\theta+\theta+C.$

10. $\log\sqrt[3]{\dfrac{(x-4)^{10}}{x-1}}+C.$

11. $\frac{1}{3}x^3\operatorname{arc\,cos}x-\frac{1}{3}\sqrt{1-x^2}+\frac{1}{9}(1-x^2)^{\frac{3}{2}}+C.$

12. $\frac{1}{2}e^{\operatorname{arc\,tan}x}\dfrac{1+x}{\sqrt{1+x^2}}+C.$

13. $\frac{1}{8}e^{2x}(1+2\sin x\cos x+2\cos^2 x)+C.$

14. $\dfrac{1}{2}\left[\sqrt{a^4-x^4}-a^2\operatorname{arc\,cos}\dfrac{x^2}{a^2}\right]+C.$

21. $\dfrac{1}{2\sqrt{2}}\log\dfrac{\sqrt{1+x^2}+\sqrt{2}\,x}{\sqrt{1+x^2}-\sqrt{2}\,x}+C.$

22. $\dfrac{1}{2(a^2-b^2)}\log\dfrac{x^2+b^2}{x^2+a^2}+C.$

23. $(a+bx^2)^{\frac{3}{2}}\left(\dfrac{x^2}{5b}-\dfrac{2a}{15b^2}\right)+C.$

25. $\pi ab.$

26. $\frac{3}{4}\pi ab.$

27. $3\pi a^2.$

28. $10\log 10-9.$

29. $\dfrac{8(b-a)^{\frac{5}{2}}}{15\sqrt{m}}.$

30. (a) $\sqrt{e^4+1}+2-\sqrt{2}+\log\dfrac{1+\sqrt{2}}{1+\sqrt{e^4+1}}.$

(b) $\sqrt{37}-\sqrt{2}+\log\dfrac{6(1+\sqrt{2})}{1+\sqrt{37}}.$ (c) $2\sqrt{3}.$

31. $\dfrac{a}{2}[\pi\sqrt{1+\pi^2}+\log(\pi+\sqrt{1+\pi})].$

32. $2a\left[\sqrt{5}-2-\sqrt{3}\log\dfrac{\sqrt{3}+\sqrt{5}}{2\sqrt{2}+\sqrt{6}}\right].$

33. $\frac{3}{2}\pi a.$

34. $12a.$

35. $\frac{4}{3}\pi ab^2.$

36. $5\pi^2a^3.$

37. $6\pi^3a^3.$

38. $\frac{1}{2}\pi^2.$

39. $\frac{64}{3}\pi a^3.$

40. $\frac{4}{3}a^2h.$

41. $2\pi b\left[b+\dfrac{a^2}{\sqrt{a^2-b^2}}\operatorname{arc\,cos}\dfrac{b}{a}\right],\ 2\pi a\left[a+\dfrac{b^2}{\sqrt{b^2-a^2}}\operatorname{arc\,cos}\dfrac{a}{b}\right].$

42. $\dfrac{128}{5}\pi a^2.$ 43. $\dfrac{\pi a}{4}.$ 44. $\dfrac{4a}{\pi}.$ 45. $BT\log\dfrac{p_1}{p_2}-\dfrac{C(n-1)}{n}(p_1{}^n-p_2{}^n)$

46. $v=\sqrt{\dfrac{2k}{s_0}}\,\dfrac{\sqrt{s_0s-s^2}}{s};\ t=\sqrt{\dfrac{s_0}{2k}}\left[\sqrt{s_0s-s^2}-\dfrac{s_0}{2}\left(\operatorname{arc\,cos}\dfrac{s_0-2s}{s_0}-\dfrac{\pi}{2}\right)\right].$

Art. 123. Pages 235–236.

1. $2\,xy^5$; $5\,x^2y^4$.
2. $\cos x \cos y$; $-\sin x \sin y$.
3. $ye^x + e^y$; $e^x + xe^y$.
4. $4\,x^3 - 2\,axy + by^2$; $-ax^2 + 2\,bxy + 4\,y^3$.
5. $y^x \log y$; xy^{x-1}.
6. $-\dfrac{1}{\sqrt{y^2-x^2}}$; $\dfrac{x}{y\sqrt{y^2-x^2}}$.
7. $y\cos x + \sin y$; $\sin x + x\cos y$.
8. $\cos(x+y)$; $\cos(x+y)$.
10. $e^x \log yz$; $\dfrac{e^x}{y}$; $\dfrac{e^x}{z}$.
11. $\cos x \cos y - \sin x \sin z$; $\cos y \cos z - \sin x \sin y$; $\cos x \cos z - \sin y \sin z$.
12. $\dfrac{1}{x+y}$; $\dfrac{1}{x+y}$; $-\dfrac{1}{z}$.
13. $\dfrac{\partial u}{\partial x} = \dfrac{2\,x^3 - y^3 - z^3}{x^2yz}$.
14. $\dfrac{2\,\pi rh}{3}$; $\dfrac{\pi r^2}{3}$.
18. 0; $\dfrac{1}{y}$; $\dfrac{1}{y}$; $-\dfrac{x}{y^2}$.
19. $m(m-1)x^{m-2}y^n$; $mnx^{m-1}y^{n-1}$; $mnx^{m-1}y^{n-1}$; $n(n-1)x^my^{n-2}$.
20. $6\,x$; a; a; $-6\,y$.
21. e^{x+y}; e^{x+y}; e^{x+y}; e^{x+y}.
22. 0; $-3\,y^2$; $-3\,y^2$; $-6\,xy$.

Art. 125. Page 241.

1. $2(x + \cos x e^{2\sin x})$.
2. $\dfrac{e^x(x-1)}{x^2+e^{2x}}$.
3. $\dfrac{1}{\sqrt{x^2+a^2}}$.
4. $e^{\frac{1}{x}}(2\,x\cos x - \cos x - x^2 \sin x)$.
5. $e^{ax}\sin x(a^2+1)$.
6. Increases $3\frac{1}{2}$ sq. units per sec.
7. $\dfrac{679}{15\sqrt{11}}\,\dfrac{\text{units}}{\text{sec}}$.
8. $-0.284\,\dfrac{\text{cu. ft.}}{\text{sec}}$.
9. $\dfrac{C(1-n)}{Rv^n}$; $\dfrac{n-1}{Rn}\sqrt[n]{\dfrac{C}{p}}$.

Art. 126. Pages 243, 244.

1. $e^xy^2\,dx + 2\,e^xy\,dy$.
2. $y^x \log y\,dx + xy^{x-1}\,dy$.
3. $\cos x \cos y\,dx - \sin x \sin y\,dy$.
4. $\dfrac{2\,x\,dx}{y^3} - \dfrac{3\,x^2\,dy}{y^4}$.
5. $a^xe^y(\log a\,dx + dy)$.
6. $\dfrac{y\,dx - x\,dy}{x^2+y^2}$.
7. $3(x^2y^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{5}{2}}y^2)dx + \frac{1}{2}(x^3y^{-\frac{1}{2}} - 4\,x^{-\frac{3}{2}}y)dy$.
8. $(y\,dx - x\,dy)\left(\dfrac{1}{x^2+y^2} + \dfrac{1}{x\sqrt{x^2-y^2}}\right)$.
10. $z^{xy}\log z(y\,dx + x\,dy) + xy\,z^{xy-1}\,dz$.
11. $\cos x \cos y \tan z\,dx - \sin x \sin y \tan z\,dy + \sin x \cos y \sec^2 z\,dz$.
12. $\dfrac{zy\,dx + zx\,dy - xy\,dz}{z^2 + x^2y^2}$.
13. $\dfrac{2\,xz\,dx + x^2\,dz}{a^3 - y^3} + \dfrac{3\,x^2y^2z\,dy}{(a^3-y^3)^2}$.
14. $dp = R\left(\dfrac{dT}{v} - \dfrac{T\,dv}{v^2}\right)$.
15. $dk = C(T^n\,dp + npT^{n-1}\,dT)$.
18. Approx., 0.00117; actual, 0.00116.
20. $\dfrac{b\,\Delta a + a\,\Delta b}{ab}$.

Art. 127. Page 245.

1. $\dfrac{6xy - y^2}{3y^2 - 3x^2 + 2xy}$.
2. $\dfrac{3x^2 \sin y + y^3 \sin x}{3y^2 \cos x - x^3 \cos y}$.
3. $-\dfrac{y}{nx}$.
4. $-\dfrac{4x(x^2+y^2)+y^2}{4y(x^2+y^2)+2y(x-2a)}$.
5. $\dfrac{4x^3 - 3y^2}{6y(x-y)}$.
6. $-\tan y$.
7. $\dfrac{pn}{b-v}$.
8. $\dfrac{\rho \sin\theta}{2\cos\theta - 3\rho}$.
9. $-\dfrac{b^2x}{a^2y}$.

Art. 128. Page 251.

1. Exact. $e^x \sin y$.
2. Exact. pv^n.
5. Exact. $\frac{1}{3}x^3 - xy$.
6. Exact. $xy^2 - (x^2+x)y$.
8. Exact. $x \sin y - e^x y$.
11. (*a*) -4; (*b*) $+4$; (*c*) -8.

Miscellaneous Examples. Pages 251–253.

1. $2xy^3z^5$; $3x^2y^2z^5$; $5x^2y^3z^4$.
2. $\dfrac{x^2}{(x^3+y^3+z^3)^{\frac{2}{3}}}$; $\dfrac{y^2}{(x^3+y^3+z^3)^{\frac{2}{3}}}$; $\dfrac{z^2}{(x^3+y^3+z^3)^{\frac{2}{3}}}$.
3. $3x^2e^{2y}\cos z$; $2x^3e^{2y}\cos z$; $-x^3e^{2y}\sin z$.
4. $\dfrac{z}{(x+y)^2+z^2}$, $\dfrac{z}{(x+y)^2+z^2}$, $-\dfrac{x+y}{(x+y)^2+z^2}$.
5. $\dfrac{4y^4 - 3x^2y^2}{2x^3y - 16xy^3 + 15y^4}$.
6. $\dfrac{b^2x}{a^2y}$.
7. $-\dfrac{ax+hy+e}{by+hx+f}$.
8. $\dfrac{3x^2y}{5y^3 - 2x^3}$.
9. $\dfrac{x + x^3 + \arctan x}{(1+x^2)\sqrt{x^2 + (\arctan x)^2}}$.
10. $3x^2e^x \arcsin x + x^3e^x \arcsin x + \dfrac{x^3e^x}{\sqrt{1-x^2}}$.
11. $\dfrac{e^{-x}(\sin x - \cos x)}{e^{-2x} + \cos^2 x}$.
12. $e^{ax}\sin x(a^2+1)$.
13. 0; -10; $72y$.
14. $-e^x\cos y$; $-e^x \sin y$; $e^x \sin y$.
15. $x^2y\sin xy - 2x\cos xy$; $xy^2 \sin xy - 2y\cos xy$; $x^3 \sin xy$.
19. $\dfrac{B}{v-b}$; $\dfrac{2a}{v^3} - \dfrac{BT}{(v-b)^2}$.
20. $-\dfrac{a^2 - b^2 + c^2}{c\sqrt{4b^2c^2 - (b^2+c^2-a^2)^2}}$.
21. $\mp\dfrac{20}{3\sqrt{11}}$; $\pm\dfrac{18}{5\sqrt{11}}$.
26. $\sqrt{3}$; 20.
27. $\Delta V = \pi r^2 \Delta h + 2\pi rh\,\Delta r$; $\dfrac{\Delta h}{h} + \dfrac{2\Delta r}{r}$.
28. (*a*) $\frac{1}{3}xy(x^2+y^2) + \phi(x) + \psi(y)$. (*b*) $\frac{1}{4}x^4 \sin y + F(x) + f(y)$.
30. (*a*) $x^2y - \frac{1}{3}y^3$. (*b*) $(1+x^2)\arctan y - y$.
31. (*a*) $\alpha T + \dfrac{\beta}{2}T^2 - \dfrac{Am(n+1)}{T^n}p\left(1 + \dfrac{a}{2}p\right) + i_0$.

(*b*) $\alpha \log T + \beta T - AB\log p - \dfrac{Amn}{T^{n+1}}p\left(1+\dfrac{a}{2}p\right) + s_0$.

Art. 129. Pages 256, 257.

1. $\frac{1}{6}x^3y^2 + F(x) + \psi(y)$.
2. $-e^x\cos y + F(x) + \psi(y)$.
3. $14,387\frac{5}{8}$.
4. $\frac{\pi}{8}$.
5. $a^3\left(2 - \frac{5\pi}{8}\right)$.
6. $\frac{a^2(8-\pi)}{8}$.
7. $\frac{5}{8}\pi a^4$.
8. $\frac{\pi a^5 c^2(4 + c^2)}{80}$.
9. $\frac{2}{3}\sqrt{2g}(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}})b$.
10. $\frac{\pi a^2}{2}$.
11. $\frac{a^3b^2}{15}$.
12. $\frac{4a^{n+2}}{2n+3}$.

Art. 130. Page 259.

3. $21\frac{1}{3}$.
4. $a^2 \arcsin\sqrt{1 - \frac{b^2}{a^2}} - b\sqrt{a^2 - b^2}$.
5. $\frac{a^2}{6}(3\pi - 8)$.
6. 12.
7. 3.549.
8. 0.049.
9. 60.1+.
10. πab.

Art. 131. Page 261.

2. $\frac{23\pi a^2}{4}$. 3. $\pi(r_1^2 - r_2^2)$. 4. $6\pi a^2$. 5. $\frac{a^2}{2}$. 7. $4.8584a^2$; $11.1416a^2$.

Art. 132. Page 264.

4. $\frac{4\pi abc}{3}$. 5. $\frac{\pi a^2 c}{2}$. 7. $\frac{4\pi a^3}{35}$. 8. $\frac{2}{3}a^3 \tan\beta$.

Art. 133. Page 267.

1. $\frac{4}{3}\pi r^3$. 2. $\frac{a^3}{360}$. 3. $\frac{9a^3}{2}$.
4. $\pi a^3\left(\frac{\log(\sqrt{2}+1)}{2\sqrt{2}} - \frac{1}{6}\right)$.
5. $\frac{2}{3}\pi a^3(\log 8 - 2)$.
6. $\frac{2\pi}{3}\left[a^2(a+b) + (a+b)^3 - \frac{1}{8}\frac{b^4}{a}\right]$.

Art. 134. Pages 270–271.

1. $\frac{\pi}{6}[c^3 - (c^2 - r^2)^{\frac{3}{2}}]$.
2. $\frac{1}{2}\pi abc$.
3. $\frac{1}{4}abc$.
6. $\frac{2}{3}\pi a^3 - \frac{8}{9}a^3$.
7. $\frac{\pi\sqrt{2}}{16}a^3$.

Art. 135. Page 273.

1. $\frac{2}{3}ka^3$; $\bar{\gamma} = \frac{4ka}{3\pi}$, if $\gamma = ky$.
2. $\frac{\pi}{3}ka^3$; $\frac{2}{3}ka$.
3. $\frac{1}{2}kl$.
4. $\frac{1}{4}\pi ka^4$; $\frac{8}{3}ka$.
5. $\frac{1}{6}kab^2$; $\frac{1}{3}kb$.
6. $\frac{3}{4}$ of density at base.

Art. 137. Pages 279, 280.

1. (a) $\left(\frac{2r}{\pi}, 0\right)$. (b) $\left(0, \frac{4r}{3\pi}\right)$.

2. (a) $\left(\frac{a[6\sqrt{2}+\log(3-2\sqrt{2})]}{8[\sqrt{2}+\log(\sqrt{2}+1)]}, \frac{4a[2\sqrt{2}-1]}{3[\sqrt{2}+\log(\sqrt{2}+1)]}\right)$. (b) $\left(\frac{3a}{5}, \frac{3a}{4}\right)$.

3. (a) $\left(\frac{2a}{e+1}, \frac{a(e^2+4-e^{-2})}{4(e-e^{-1})}\right)$. (b) $\left(\frac{2a}{e+1}, \frac{a(e^2+4-e^{-2})}{8(e-e^{-1})}\right)$.

4. (a) $\left(\pi a, \frac{4a}{3}\right)$. (b) $\left(\pi a, \frac{5a}{6}\right)$.

5. $\left(\frac{5a}{6}, 0\right)$.

6. $\left(\frac{5a}{7}, 0\right)$.

7. $\left(\frac{3\pi a}{16}, 0\right)$.

8. $\left(\frac{2a}{3}, 0\right)$.

9. $\left(0, 0, \frac{8r}{15}\right)$.

10. $\bar{x} = \bar{y} = \frac{128a}{105\pi}$.

11. $\bar{x} = \bar{y} = z = \frac{2}{5}a$.

13. $1\frac{7}{12}$ in. from base of cylinder.

14. Take mass m_1 at origin and X-axis along the side containing m_1 and m_2; then $\bar{x} = \frac{13}{24}a$, $\bar{y} = \frac{5\sqrt{3}}{24}a$.

Art. 138. Page 282.

1. $\frac{1}{3}bh^3$; $\frac{1}{3}h^2$.
2. $\frac{1}{12}bh^3$; $\frac{1}{12}h^2$.
3. $\frac{1}{2}\pi a^4$; $\frac{1}{2}a^2$.
4. $\frac{2}{5}\pi ka^5$; $\frac{3}{5}a^2$.
5. $\frac{1}{4}bh^3$; $\frac{1}{2}h^2$.
6. $\frac{1}{2}\pi a^3$; $\frac{1}{2}a^2$.
7. $\frac{35}{12}\pi a^4$; $\frac{35}{36}a^2$.
8. $\frac{256}{15}a^3$; $\frac{32}{15}a^2$.
9. $\frac{4}{15}\pi a^5$; $\frac{1}{3}a^2$.
10. $\frac{1}{6}\pi ka^6$; $\frac{1}{3}a^2$.

Art. 139. Pages 285, 286.

The squares of the radii of gyration are :

1. (a) $\frac{1}{6}a^2$; (b) $\frac{1}{12}a^2$.

2. (a) $\frac{b^2}{4}$; (b) $\frac{a^2}{4}$;

   (c) $\frac{5}{4}b^2$; (d) $\frac{a^2+b^2}{4}$.

3. $\frac{1}{4}(a_1^2+a_2^2)$.

4. $\frac{2}{5}a^2$.

5. $\frac{3}{10}a^2$.

6. (a) $\frac{1}{2}a^2$; (b) $\frac{3}{2}a^2$; (c) $\frac{a^2}{4}+\frac{h^2}{3}$.

8. $\bar{y} = \frac{b^2+at-t^2}{2(a+b-t)}$.

9. (a) $k^2 = \frac{b^3+(a-t)t^2}{3(a+b-t)}$.

   (b) $k_g^2 = k^2 - \bar{y}^2$.

Art. 140. Pages 287, 288.

1. $\frac{1}{2}kb(h_2^2-h_1^2)$.
2. $\frac{1}{3}kbh^2$.
5. $\frac{4}{15}\sqrt{2g}bh^{\frac{3}{2}}$.
6. $0.54\sqrt{2g\pi}a^{\frac{5}{2}}$.
9. $\frac{2Ah}{a\sqrt{2gh}}$.
10. $\frac{14\pi r^{\frac{5}{2}}}{15a\sqrt{2g}}$.

Miscellaneous Examples. Pages 288–290.

1. 12.022.
2. $\frac{3}{8}\pi ab$.
3. $\frac{3}{2}a^2$.
4. $\frac{1}{8}\pi a^2$.
5. $\dfrac{a^6(27\sqrt{3}+10\pi)}{64}$.
6. $\frac{8}{3}a^2$.
7. $\dfrac{\pi abr^2}{c}$.
8. $\dfrac{4\pi m^5}{ab}$.
9. $\dfrac{8\pi n^3}{3}$.
10. $\dfrac{2\pi a}{\sqrt{mn}}$.
12. $m\dfrac{\sqrt{a^2b^2+1}}{a^2}$.
13. $\frac{3}{2}ac+\frac{3}{4}(a^2+4c^2)\text{arc tan}\dfrac{a}{2c}$.
14. $\dfrac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}$.
15. $\dfrac{a^2c}{2}\left(\dfrac{3}{2}+\log\dfrac{b^2}{ac}\right)$.
16. $\frac{1}{4}$ of density at vertex.
17. $\frac{5}{8}$ of density at base.
18. $2\pi^2a^2b$; $4\pi^2ab$.
19. $\dfrac{a^2}{\sqrt{3}}\sqrt{\dfrac{4a^2}{3}+4\pi^2\left(r+\dfrac{a}{3}\right)^2}$.
23. $k^2=\dfrac{b_1h_1^3-b_2h_2^3}{12(b_1h_1-b_2h_2)}$.
24. 10.814 in.
25. $c=\dfrac{2}{\pi}\left(R+\dfrac{r_1^2+r_2^2}{4R}\right)$.
29. $\bar{x}=\frac{1}{3}OC$, $\bar{y}=\frac{2}{9}OD$, $\bar{z}=\frac{1}{3}OA$.

Art. 144. Page 298.

1. $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\cdots,\quad -\infty<x<\infty$.
2. $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dfrac{x^8}{8!}+\cdots,\quad -\infty<x<\infty$.
3. $1+x\log a+\dfrac{x^2(\log a)^2}{2!}+\dfrac{x^3(\log a)^3}{3!}+\cdots,\quad -\infty<x<\infty$.
4. $1+x+\dfrac{x^2}{2}-\dfrac{x^4}{8}+\cdots,\quad -\infty<x<\infty$.
5. $\log 2+\dfrac{x}{2}+\dfrac{x^2}{8}+\dfrac{x^4}{2^8\cdot 4!}\cdots,\quad -\infty<x<\infty$.
6. $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\dfrac{x^5}{5}-\cdots,\quad -1<x\leqq 1$.
7. $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5}+\cdots,\quad -1<x<1$.
8. $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\dfrac{x^9}{9}-\cdots,\quad -1\leqq x\leqq 1$.
9. $1-n\dfrac{\theta^2}{2!}+(3n^2-2n)\dfrac{\theta^4}{4!}-\cdots,\quad -\infty<\theta<\infty$.
10. $1+\theta+\theta^2+\dfrac{2\theta^3}{3}+\dfrac{\theta^4}{2}+\cdots,\quad -\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$.
11. $1+\dfrac{\theta^2}{2}+\dfrac{5\theta^4}{24}+\dfrac{61\theta^6}{720}+\dfrac{277\theta^8}{8064}+\cdots,\quad -\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$.

12. $1+\theta+\frac{{}^2\theta}{2!}+\frac{2\,\theta^3}{3!}+\frac{5\,\theta^4}{4!}+\cdots, \quad -1 \leqq \theta \leqq 1.$

13. $x-\frac{x^3}{3!}+\frac{9\,x^5}{5!}-\cdots, \quad -\infty < x < \infty.$

14. $-\left(\frac{\theta^2}{2}+\frac{\theta^4}{12}+\frac{\theta^6}{45}+\frac{17\,\theta^8}{2520}+\cdots\right), \quad 0 \leqq \theta < \frac{\pi}{2}.$

15. $1+\frac{x^2}{2\,!}+\frac{x^4}{4\,!}+\frac{x^6}{6\,!}+\cdots, \quad -\infty < x < \infty.$

16. $-x+\frac{x^2}{2}+\frac{2\,x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\cdots, \quad -\infty < x < \infty.$

Art. 145. Page 301.

1. $e^x\left(1+h+\frac{h^2}{2\,!}+\frac{h^3}{3\,!}+\cdots\right).$

2. $x^m+mx^{m-1}y+\frac{m(m-1)}{2\,!}x^{m-2}y^2+\cdots.$

3. $x^6+6\,x^5y+15\,x^4y^2+20\,x^3y^3+15\,x^2y^4+6\,xy^5+y^6.$

4. $\text{arc}\sin x+\frac{h}{(1-x^2)^{\frac{1}{2}}}+\frac{h^2x}{2\,!\,(1-x^2)^{\frac{3}{2}}}+\frac{h^3(1+2\,x^2)}{3\,!\,(1-x^2)^{\frac{5}{2}}}+\cdots$

5. $\log\sin x+h\cot x-\frac{h^2}{2}\csc^2 x+\frac{h^3}{3}\csc^2 x\cot x-\cdots.$

6. $x^3+9\,x^2+23\,x+22.$ 7. $x^2+5\,x-11.$

9. $\sin a+(x-a)\cos a-\frac{(x-a)^2}{2\,!}\sin a-\frac{(x-a)^3}{3\,!}\cos a+\cdots.$

10. $5(x-1)^3+9(x-1)^2+4(x-1)-10;$
$5(x-3)^3+39(x-3)^2+100(x-3)+74.$

11. $(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots, \quad 0 < x \leqq 2.$

12. $\frac{1}{a}-\frac{x-a}{a^2}+\frac{(x-a)^2}{a^3}-\frac{(x-a)^3}{a^4}+\cdots, \quad 0 < x < 2\,a.$

Art. 147. Page 305.

7. 0.52360. 8. 1.31105. 9. $2\,\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right).$

10. $\frac{303}{256}x+\frac{47}{512}\sin 2\,x+\frac{5}{128}\sin x\cos^3 x+\cdots.$

11. 0.746824.

Art. 149. Page 311.

2. 0.21256.

3. 11.008.

4. 3.433987 ; 4.290459.

5. 0.009804 ; 0.010309.

6. 0.016+.

7. $-19°\,15' < x < 19°\,15'.$

8. Error $< \frac{m^2}{2\,!}+\frac{m^3}{3\,!}.$

9. (*a*) 10.472. (*b*) 10.50. Relative error = 0.0027.

Art. 150. Page 313.

1. Min. if $x = \frac{\pi}{4} + n\pi$.
2. No max. nor min.
3. Max. if $x = \frac{\pi}{3}$; min. if $x = -\frac{\pi}{3}$.
4. Min. if $x = \frac{1}{2a} \log \frac{q}{p}$.
5. Max. if $x = -\tan x$, $\frac{\pi}{2} < x < \pi$.
   Min. if $x = -\tan x$, $\frac{3\pi}{2} < x < 2\pi$.
6. Max. if $\theta = 0$; min. if $\theta = \pi$, or arc cos $\sqrt[3]{\frac{2}{3}}$.
7. Min. if $x = 0$.
8. Min. if $x = 1$.
9. Max. if $x = 0$.
10. Max. if $x = 0$.

Art. 151. Page 317.

1. 2.
2. $\frac{1}{2}$.
3. $\frac{1}{2}$.
4. $\log \frac{a}{b}$.
5. 0.
6. 0.
7. 2.
8. -2.
9. $\frac{1}{3}$.
10. $\log a$.
11. ∞.
12. 0.
13. 0.
14. 1.
15. 1.
16. 1.
17. e^2.
18. 0.
19. $\frac{1}{3}$.
20. $-\frac{1}{3}$.
21. 1.
22. ∞.
23. $\frac{7}{6}$.
24. $\frac{1}{2}$.

Art. 152. Pages 319, 320.

1. Triple point at origin; $\frac{dy}{dx} = 0, 2, -2$.
2. Conjugate point at origin.
3. Triple point at origin.
4. Cusp at origin.
5. Triple point at (0, 2).

Miscellaneous Exercises. Pages 320, 321.

1. (*a*) Convergent if $x < \frac{3}{2}$. (*b*) Convergent if $-\infty < x < \infty$.
   (*c*) Convergent if $-\infty < x < \infty$. (*d*) Convergent if $x \neq 0$.
2. $\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \cdots$.
3. $1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{8x^7}{7!} + \cdots$.
4. $1 - \frac{nx^2}{2!} + (3n^2 - 2n)\frac{x^4}{4!} - \cdots$.
5. $1 - \frac{x}{2} + \frac{x^2}{12} + \cdots$.
6. $2 + \frac{x^2}{4} + \frac{x^4}{96} + \frac{x^6}{1440} + \cdots$.

9. $275 \pm 85h + h^2 \pm h^3$;
   249.617; 258.048; 266.511; 283.509; 292.032; 300.563.

12. $x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \frac{16}{35}x^7 + \cdots$.
14. 1.
15. $\frac{1}{6}$.
16. 2.

21. $1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \cdots$.
22. $x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots$.
23. $1 \pm x - \frac{x^2}{2} \mp \frac{x^3}{6} + \frac{x^4}{24} \pm \cdots$.
24. $1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots$.

INDEX

(Numbers refer to pages.)